Inductive Characterization of the Cumulative Hierarchy

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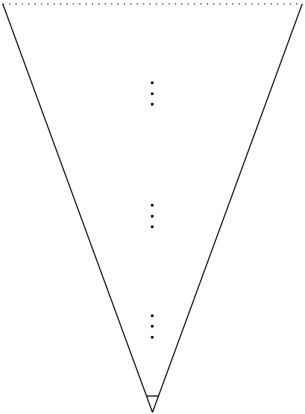
Outline

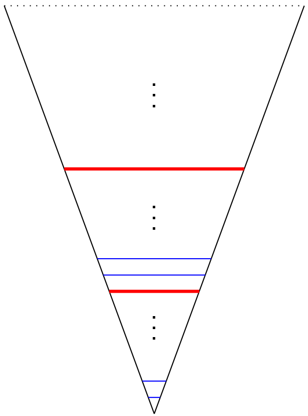
Motivation

The universe of ZF sets is rich in structure but unrealizable...



Motivation

Stratify it into cumulative slices and stages which are all realizable!



Common Definition

We define the class $\mathcal{V}$ by transfinite recursion on ordinals:

$$\mathcal{V}_\alpha := \mathcal{P}^{\,\alpha}(\emptyset) \qquad \mathcal{V} := \bigcup_{\alpha \in \mathcal{O}} \mathcal{V}_\alpha$$

- Hierarchy $\mathcal{V}_\alpha$ inherits key properties from $\mathcal{O}$ (well-ordering)
- Definition depends on ordinal theory, transfinite recursion etc.

$\Rightarrow$ Appears late in a typical first-order presentation of ZF

Inductive Definition

We define the class $\mathcal{Z}$ by two (almost) disjoint inductive rules:

$$\frac{x \in \mathcal{Z}}{\mathcal{P}(x) \in \mathcal{Z}} \qquad\qquad \frac{M \subseteq \mathcal{Z}}{\bigcup M \in \mathcal{Z}}$$

- Definition depends only on ZF axioms, no background needed
- Establishing key properties of $\mathcal{Z}$ is not trivial

$\Rightarrow$ There is some work for a RIL...

Goals

The three key properties of $\mathcal{Z}$ are:

1. $\mathcal{Z}$ is well-ordered by $\subseteq$
2. $\mathcal{Z}$ exhausts the (well-founded) sets
3. $\mathcal{Z}$ is (order-)isomorphic to the class $\mathcal{O}$ of ordinals

Once they have been established, we have:

- The isomorphy of $\mathcal{Z}$ and the $\mathcal{V}_\alpha$ hierarchy
- Inner models for (subtheories) of ZF

Related Work

During the RIL, two related papers were published:

- "Transfinite Constructions in Classical Type Theory" by Gert Smolka, Steven Schäfer and Christian Doczkal [1]
  - General inductive towers on type level
  - Well-ordering proof with few requirements
- "Axiomatic Set Theory in Type Theory" by Gert Smolka [2]
  - Full presentation of $\mathcal{O}$ and $\mathcal{Z}$ in inductive shape
  - Profound theory of well-orderings

Definitions

Let X be a type and $\leq$ a binary relation on X.

- $\mathcal{D}(\leq) := \{x \mid \exists y.\, x \leq y\}$
- $\leq$ is *reflexive* iff $x \leq y \Rightarrow x \leq x \wedge y \leq y$
- $\leq$ is *antisymmetric* iff $x \leq y \Rightarrow y \leq x \Rightarrow x = y$
- $\leq$ is *transitive* iff $x \leq y \Rightarrow y \leq z \Rightarrow x \leq z$
- $\leq$ is *linear* iff $x, y \in \mathcal{D}(\leq) \Rightarrow x \leq y \vee y \leq x$

We say $\leq$ is a *partial ordering* iff $\leq$ reflexive, antisymmetric and transitive. A partial ordering that is linear is called a *linear ordering*.

Definitions (ctd.)

Let X be a type and $\leq$ a binary relation on X.

- $\mathcal{L}_p^{\leq} := \{x \in p \mid \forall y \in p.\, x \leq y\}$
- $\Sigma^{\leq} := \{p \mid p \subseteq \mathcal{D}(\leq) \wedge \forall x\, y.\, x \leq y \Rightarrow y \in p \Rightarrow x \in p\}$
- $\Sigma_x^{\leq} := \{y \mid y \leq x \wedge x \neq y\}$
- $\leq\!|_p := \lambda x\, y.\, x \leq y \wedge x \in p \wedge y \in p$

We say $\leq$ is a *well-ordering* iff $\leq$ is a partial ordering and $\mathcal{L}_p^{\leq}$ is inhabited whenever $p \subseteq \mathcal{D}(\leq)$ is inhabited. Clearly, well-orderings are linear and restrictions of well-orderings are well-orderings again.

Similarities

Let $\leq_1$ be a relation on X and $\leq_2$ be a relation on Y.
A relation U from X to Y is:

- *simulative* iff $U\,x\,y \Rightarrow x' \leq_1 x \Rightarrow \exists y'.\, U\,x'\,y' \wedge y' \leq_2 y$
- a *simulation* iff $\mathcal{D}(U) \subseteq \mathcal{D}(\leq_1)$ and U is simulative
- a *similarity* iff U and U^{-1} are functional simulations
- an *isomorphism* iff also $\mathcal{D}(U) \equiv \mathcal{D}(\leq_1)$ and $\mathcal{D}(U^{-1}) \equiv \mathcal{D}(\leq_2)$

Clearly, $D(U)$ and $D(U^{-1})$ are segments of the respective orderings. Moreover, if U and V are similarities we have $U \subseteq V$ or $V \subseteq U$. Thus similarities are stable under union.

Embedding Theorem

Let U be a similarity from X to Y.
We say U is *maximal* if $\mathcal{D}(U) \supseteq \mathcal{D}(\leq_1)$ or $\mathcal{D}(U^{-1}) \supseteq \mathcal{D}(\leq_2)$.

Theorem 1

Let $\leq_1$ be a well-ordering on X and $\leq_2$ be a well-ordering on Y. Then there exists a unique maximal similarity from X to Y.

$\Rightarrow$ Given two well-orderings $\leq_1$ and $\leq_2$,
$\leq_1$ is isomorphic to a section of $\leq_2$ or
$\leq_2$ is isomorphic to a section of $\leq_1$.

Basic Set Theory

We assume:

1. a type $\mathcal{S}$ of sets with a binary relation $\in$.
2. sets to be extensional $(x \equiv y \Rightarrow x = y)$
3. axioms for $\emptyset, \{x, y\}, \bigcup x$ and $\mathcal{P}(x)$
4. replacement: $z \in R[x]$ iff there exists $y \in x$ such that $R\,y\,z$
   (and $R\,y$ is unique)

- A class p is *realizable* iff $p \equiv x$ for some set x.
- A set x is *transitive* $(x \in \mathcal{T})$ iff $y \subseteq x$ for all $y \in x$.
- A set x is *well-founded* $(x \in \mathcal{W})$ iff $y \in \mathcal{W}$ for all $y \in x$.

Orderings on Sets

We say a relation $\leq$ on $\mathcal{S}$ is *realizable* iff $\mathcal{D}(\leq)$ is realizable.
Moreover, $\leq$ is *complete* iff $\Sigma_x^{\leq}$ are realizable but $\leq$ itself is not.

Lemma 2

Complete well-orderings are isomorphic.

We define the *inclusion ordering* of a class p with $I_p := \subseteq|_p$.
Clearly, I_p is a partial ordering and Σ_x^p are realizable.
Thus I_p is complete iff p is unrealizable.

Tower Construction

Let f be a function from $\mathcal{S}$ to $\mathcal{S}$. We define the tower T for f:

$$\frac{x \in T}{f(x) \in T} \qquad \frac{M \subseteq T}{\bigcup M \in T}$$

If f is *increasing* ($x \subseteq f(x)$), *cumulative* ($x \in f(x)$) and preserves transitivity and well-foundedness, all the following hold:

- I_T is a linear ordering.
- $T \subseteq \mathcal{T}$ and $T \subseteq \mathcal{W}$.
- T is unrealizable.
- I_T is a complete well-ordering.

Generic Results

Consider $f(x) := x \cup \mathcal{P}(x)$. For $x \in \mathcal{T}$ we have $x \subseteq \mathcal{P}(x)$.
Since $\mathcal{P}$ and f both preserve transitivity, we have $\mathcal{Z} \equiv T_f$.
$\Rightarrow$ Goal 1 ✓

Moreover, consider $g(x) := x \cup \{x\}$. We define $\mathcal{O} := T_g$.
We can apply Lemma 2 for the complete well-orderings $I_{\mathcal{Z}}$ and $I_{\mathcal{O}}$.
$\Rightarrow$ Goal 3 ✓

Goal 2

Theorem 3

Every well-founded set $x \in \mathcal{W}$ is member of some $y \in \mathcal{Z}$.

We have two proofs (induction on $x \in \mathcal{W}$):

1.
   - Refine the statement to the least cumulative set y with $x \in y$.
   - The IH is that there exists such a set y' for all $x' \in x$.
   - Obtain y as the power set of the union of all y' (replacement).
2.
   - The IH is that there exists $y' \in \mathcal{Z}$ with $x' \in y'$ for all $x' \in x$.
   - Hence there exists a least such y' for all $x' \in x$ (well-ordering)
   - Obtain y as the power set of the union of all y' (replacement).

$\Rightarrow$ Goal 2 ✓

On the Isomorphism of $\mathcal{Z}$ and $\mathcal{O}$

In the first version of the above proof, we use a notable relation:

$$R\,\alpha\,x :\equiv \alpha \in \mathcal{O} \text{ and } x \in \mathcal{Z} \text{ is least with } \alpha \subseteq x$$

As a first fact, we can consider two "recursion equations":

1. $R\,\alpha\,x \Rightarrow R\,(\alpha \cup \{\alpha\})\,(\mathcal{P}(x))$
2. $x \subseteq \mathcal{O} \Rightarrow R\,(\bigcup x)\,(\bigcup R[x])$

Moreover, we have the following "partitioning" of subsets M of $\mathcal{Z}$:

$$\bigcup M \in M \text{ xor } M \subseteq \bigcup M \text{ (by linearity of } \mathcal{Z})$$

This implies that we have $\alpha \notin x$ whenever $R\,\alpha\,x$.

On the Isomorphism of $\mathcal{Z}$ and $\mathcal{O}$ (ctd.)

All the following hold:

1. R is total on $\mathcal{O}$ (by Theorem 3)
2. R is surjective on $\mathcal{Z}$ (by recursion equations)
3. R is functional (by extensionality)
4. R is injective (by partitioning)
5. R respects $I_{\mathcal{O}}$ (by linearity of $I_{\mathcal{O}}$)
6. R respects $I_{\mathcal{Z}}$ (by linearity of $I_{\mathcal{Z}}$)

$\Rightarrow$ We have an explicit characterization of the isomorphism!

We even have a second one: $R\,\alpha\,x \Leftrightarrow \alpha \in \mathcal{O} \wedge x \in \mathcal{Z} \wedge \alpha = x \cap \mathcal{O}$

Conclusion

We conclude the RIL with the following remarks:

- Formalizing set theory in a rich type theory like CiC allows for a concise presentation, elegant (and intuitive) proofs and an interactive means for teaching.
- Especially exploring the cumulative hierarchy benefits from inductive definitions whereby the link to the common definition is kept visible (recursion equations).
- Lose ends are the search for “ordinal types”, a more informative linearity proof and exploring Gödel’s constructible universe in relation to $\mathcal{V}$.

Statistics

The RIL in numbers:

- Workload: $>$180h
- Memo 1: embedding theorem (6 pages)
- Memo 2: cumulative hierarchy (2 pages...)
- Development 1: embedding theorem (500 lines)
- Development 2: cumulative hierarchy (1800 lines)
- Tea: 150g Korean Seogwang Sencha

References

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  Introduction to Set Theory.
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