Mechanised Constructive Reverse Mathematics: Soundness and Completeness of Bi-Intuitionistic Logic

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Bi-Intuitionistic Logic

Extends intuitionistic logic with exclusion, a dual to implication:

\[ w \models \varphi \rightarrow \psi \ := \ \forall w' \geq w. \ w' \models \varphi \rightarrow w' \models \psi \]

\[ w \models \varphi \rightarrow \neg \psi \ := \ \exists w' \leq w. \ w' \models \varphi \land w' \not\models \psi \]

\[ w \models \varphi \neg \psi \ := \ \neg (\forall w' \leq w. \ w' \models \varphi \rightarrow w' \models \psi) \]

Corresponds to extending proof calculi with axioms for exclusion, e.g.

\[ \psi \lor (\top \rightarrow \psi) \]

capturing the case distinction \( w \models \varphi \lor \neg (\forall w' \leq w. \ w' \models \varphi) \).
A Case for Computer Mechanisation

- Semantics introduced by Grzegorczyk (1964)
- Henkin-style completeness proof by Klemke (1971)
- Extensive investigation by Rauszer (1980)
- Rauszer’s completeness proofs fixed for the propositional case by Goré and Shillito (2020)
- Goré and Shillito’s results mechanised in the Coq proof assistant by Shillito (2023)

Soundness and completeness are guaranteed to be correct!
Constructive Reverse Mathematics\textsuperscript{1}

In foundations like constructive type theory, many sub-classical distinctions become visible:

- **Excluded Middle (LEM)** := $\forall P : \mathbb{P}. \; P \lor \neg P$
- **Weak Excluded Middle (WLEM)** := $\forall P : \mathbb{P}. \; \neg P \lor \neg \neg P$
- **Double Negation Shift (DNS)** := $\forall p : \mathbb{N} \rightarrow \mathbb{P}. (\forall n. \neg \neg p n) \rightarrow \neg \neg (\forall n. p n)$
- **Markov’s Principle (MP)** := $\forall f : \mathbb{N} \rightarrow \mathbb{B}. \; \neg \neg (\exists n. f n = \text{true}) \rightarrow \exists n. f n = \text{true}$

Some classically valid theorems are actually equivalent to constructively weaker principles...

Correct theorems can still be analysed regarding their logical strength!

\textsuperscript{1}Ishihara (2006); Diener (2018)
Constructive Reverse Mathematics of Completeness Theorems

Does $\mathcal{T} \models \varphi$ imply $\mathcal{T} \vdash \varphi$ constructively?

Current situation in the literature on first-order logic:

- Completeness equivalent to Boolean Prime Ideal Theorem (Henkin, 1954)
- Completeness requires Markov’s Principle (Kreisel, 1962)
- Completeness equivalent to Weak König’s Lemma (Simpson, 2009)
- Completeness equivalent to Weak Fan Theorem (Krivtsov, 2015)
- Completeness holds fully constructively (Krivine, 1996)
- Systematic investigation is work in progress (Herbelin and Kirst, 2023)
Classical Outline for (Bi-)Intuitionistic Propositional Logic

Employing prime theories \((\varphi \lor \psi \in \mathcal{T} \rightarrow \varphi \in \mathcal{T} \lor \varphi \in \mathcal{T})\):

- Lindenbaum Extension: if \(\mathcal{T} \not\vdash \varphi\) then there is prime \(\mathcal{T}'\) with \(\mathcal{T}' \not\vdash \varphi\)

- Universal Model \(\mathcal{U}\): consistent prime theories related by inclusion

- Truth Lemma for \(\mathcal{T}\) in \(\mathcal{U}\): \(\varphi \in \mathcal{T} \iff \mathcal{T} \vDash \varphi\)

- Model Existence: if \(\mathcal{T} \not\vdash \varphi\) then there is \(\mathcal{M}\) with \(\mathcal{M} \vDash \mathcal{T}\) and \(\mathcal{M} \not\vDash \varphi\)

- Quasi-Completeness: if \(\mathcal{T} \vDash \varphi\) then \(\neg\neg(\mathcal{T} \vdash \varphi)\)

- Completeness: if \(\mathcal{T} \vDash \varphi\) then \(\mathcal{T} \vdash \varphi\)
Constructive Completeness Proof???

For $\mathcal{T}$ quasi-prime ($\varphi \lor \psi \in \mathcal{T} \rightarrow \neg
\neg(\varphi \in \mathcal{T} \lor \varphi \in \mathcal{T})$):

- Lindenbaum Extension: if $\mathcal{T} \not\models \varphi$ then there is quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \not\models \varphi$

- Universal Model: consistent quasi-prime theories related by inclusion

- Truth Lemma: fails immediately

- Model Existence: fails

- Quasi-Completeness: fails

- Completeness: needs MP/LEM depending on theory complexity and syntax fragment
Constructive Completeness Proof?

For $\mathcal{T}$ quasi-prime ($\varphi \lor \psi \in \mathcal{T} \rightarrow \neg\neg(\varphi \in \mathcal{T} \lor \psi \in \mathcal{T})$) and stable ($\neg\neg(\varphi \in \mathcal{T}) \rightarrow \varphi \in \mathcal{T}$):

- **Lindenbaum Extension**: if $\mathcal{T} \not\models \varphi$ then there is stable quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \not\models \varphi$

- **Universal Model**: consistent stable quasi-prime theories related by inclusion

- **Truth Lemma**: fails for disjunction

- **Model Existence**: fails

- **Quasi-Completeness**: fails

- **Completeness**: needs MP/LEM depending on theory complexity and syntax fragment
The Issue with Disjunction

Truth Lemma case for disjunctions $\varphi \lor \psi$:

$$\varphi \lor \psi \in \mathcal{T} \iff \mathcal{T} \models \varphi \lor \psi$$

$$\overset{\text{def}}{\iff} \mathcal{T} \models \varphi \lor \mathcal{T} \models \psi$$

$$\overset{\text{IH}}{\iff} \varphi \in \mathcal{T} \lor \psi \in \mathcal{T}$$

- So we really need prime theories to interpret disjunctions
- Primeness from Lindenbaum Extension is constructive no-go
Model Existence via WLEM

Weak law of excluded middle WLEM \( \equiv \forall P : \mathbb{P}. \neg P \lor \neg \neg P \)

Lemma

Assuming WLEM, every stable quasi-prime theory is prime.

Proof.

Assume \( \varphi \lor \psi \in \mathcal{T} \). Using WLEM, decide whether \( \neg(\varphi \in \mathcal{T}) \) or \( \neg \neg(\varphi \in \mathcal{T}) \). In the latter case, conclude \( \varphi \in \mathcal{T} \) directly by stability. In the former case, derive \( \psi \in \mathcal{T} \) using stability, since assuming \( \neg(\psi \in \mathcal{T}) \) on top of \( \neg(\varphi \in \mathcal{T}) \) contradicts quasi-primeness for \( \varphi \lor \psi \in \mathcal{T} \). \( \square \)

Classical proof outline works again up to Model Existence and Quasi-Completeness!
Backwards Analysis

Which logical principles are really necessary for the intermediate statements?

Fact

Model Existence implies WLEM.

Proof.

Given $P$, use model existence on $T := \{x_0 \lor \neg x_0\} \cup \{x_0 \mid P\} \cup \{\neg x_0 \mid \neg P\}$. We have $T \not\models \bot$ so if $M \models T$, then either $M \models x_0$ or $M \models \neg x_0$, so either $\neg \neg P$ or $\neg P$, respectively. \qed

Fact

Quasi-Completeness implies the following principle: $\forall p : \mathbb{N} \rightarrow \mathbb{P}. \neg \neg (\forall n. \neg p\ n \lor \neg \neg p\ n)$

Proof.

Using similar tricks for $T := \{x_n \lor \neg x_n\} \cup \{x_n \mid p\ n\} \cup \{\neg x_n \mid \neg p\ n\}$. \qed

Since Quasi-Completeness also follows from DNS, there is no hope it is equivalent to WLEM...
Weak Excluded-Middle Shift\(^2\)

\[
\text{WLEMS} := \forall p : \mathbb{N} \rightarrow \mathbb{P}. \left( \forall n. \neg \neg (\neg p \lor \neg \neg p) \right) \rightarrow \neg \neg (\forall n. \neg p \lor \neg \neg p) \\
\equiv \forall pq : \mathbb{N} \rightarrow \mathbb{P}. \left( \forall n. \neg \neg (\neg p \lor \neg q) \right) \rightarrow \neg \neg (\forall n. \neg p \lor \neg q)
\]

**Lemma**

*Assuming WLEMS, every stable quasi-prime theory is not not prime.*

**Proof.**

Assume \( \mathcal{T} \) not prime and derive a contradiction. Given the negative goal, from WLEMS we obtain \( \forall \varphi. \neg (\varphi \in \mathcal{T}) \lor \neg \neg (\varphi \in \mathcal{T}) \). This yields exactly the instances of WLEM needed to derive that \( \mathcal{T} \) is prime, contradiction. \( \square \)

Already this lemma turns out to be enough for Quasi-Completeness!

\(^2\)Mentioned in systematic study by Umezawa (1959) but absent from the literature otherwise
Quasi-Completeness via WLEMS

Refined proof outline using WLEMS:

- **Lindenbaum Extension**: if $\mathcal{T} \not\vdash \varphi$ then there is stable not not prime $\mathcal{T}'$ with $\mathcal{T}' \not\vdash \varphi$

- **Universal Model** $\mathcal{U}$: consistent stable prime theories related by inclusion

- **Truth Lemma for** $\mathcal{T}$ in $\mathcal{U}$: $\varphi \in \mathcal{T} \iff \mathcal{T} \models \varphi$

- **Quasi Model Existence**: if $\mathcal{T} \not\models \varphi$ then there not not is $\mathcal{M}$ with $\mathcal{M} \models \mathcal{T}$ and $\mathcal{M} \not\models \varphi$

- **Quasi-Completeness**: if $\mathcal{T} \models \varphi$ then $\neg \neg (\mathcal{T} \models \varphi)$

- **Completeness**: needs MP/LEM depending on theory complexity and syntax fragment
Consequences and Open Questions

Consequences:

- WLEM and Model Existence are equivalent
- WLEMS, Quasi Model Existence, and Quasi-Completeness are equivalent
- WLEMS and MP together imply completeness for enumerable contexts

Open questions:

- What restriction of WLEMS is sufficient for enumerable contexts?
- What is the relation of WLEMS to the fan theorem?
- What is the constructive status of the traditional semantics of bi-intuitionistic logic?
- What observations transport to first-order bi-intuitionistic logic (or other logics)?
Bibliography I


Working Towards an Explanation

There are multiple dimensions at play:

- Syntax fragment (e.g., propositional, minimal, negative, full)
- Complexity of the context (e.g., finite, decidable, enumerable, arbitrary)
- Cardinality of the signature (e.g., countable, uncountable)
- Representation of the semantics (e.g., Boolean, decidable, propositional)

Ongoing systematic investigation using Coq:

- Started by Herbelin and Ilik (2016) and Forster, Kirst, and Wehr (2021)
- New observations by Hagemeier and Kirst (2022) and Kirst (2022)
- Comprehensive overview of current landscape by Herbelin (2022)
- Today: syntactic disjunction, arbitrary contexts, countable signature, prop. semantics
The Case of Soundness

Some axioms of like $\psi \lor (\top \rightarrow \psi)$ are only valid in models behaving classically:

decidable models $\subseteq$ stable models $\subseteq$ axiomatic models

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Lemma

*Soundness holds for every model satisfying the (critical) axioms.*

Corollary

*Assuming LEM, soundness holds for all models.*

Fact

*Soundness for all models implies LEM.*

Proof.

For a proposition $P$ consider the single-world model with $w \models x_0$ iff $P$. By assuming soundness, we have $w \models x_0 \lor (\top \rightarrow x_0)$ which is equivalent to $P \lor \neg P$. \[\square\]
Quasi-Completeness via DNS

Assuming double-negation shift DNS := ∀X. ∀p : X → P. (∀x. ¬¬p x) → ¬¬(∀x. p x):

- **Lindenbaum Extension**: if \( T \not\vdash \varphi \) then there is stable quasi-prime \( T' \) with \( T' \not\vdash \varphi \)
- **Universal Model** \( U \): consistent stable quasi-prime theories related by inclusion
- **Pseudo Truth Lemma** for \( T \) in \( U \): \( \varphi \in T \iff \neg\neg(T \vdash \varphi) \)
- **Pseudo Model Existence**: if \( T \not\vdash \varphi \) then there is \( M \) with \( \neg\neg(M \vdash T) \) and \( M \not\vdash \varphi \)
- **Quasi-Completeness**: if \( T \vdash \varphi \) then \( \neg\neg(T \vdash \varphi) \) (also since DNS \( \leftrightarrow \neg\neg\text{LEM} \))
- **Completeness**: needs MP/LEM depending on theory complexity and syntax fragment