Mechanised Constructive Reverse Mathematics: Soundness and Completeness of Bi-Intuitionistic Logic

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Bi-Intuitionistic Logic

Extends intuitionistic logic with exclusion, a dual to implication:

$$w \Vdash \varphi \to \psi \;:=\; \forall w' \geq w.\, w' \Vdash \varphi \;\to\; w' \Vdash \psi$$
$$w \Vdash \varphi \prec \psi \;:=\; \exists w' \leq w.\, w' \Vdash \varphi \wedge w' \nVdash \psi$$
$$w \Vdash \varphi \prec \psi \;:=\; \neg(\forall w' \leq w.\, w' \Vdash \varphi \;\to\; w' \Vdash \psi)$$

Corresponds to extending proof calculi with axioms for exclusion, e.g.

$$\psi \vee (\top \prec \psi)$$

capturing the case distinction $w \Vdash \varphi \vee \neg(\forall w' \leq w.\, w' \Vdash \varphi)$.

A Case for Computer Mechanisation

- Semantics introduced by Grzegorczyk (1964)
- Henkin-style completeness proof by Klemke (1971)
- Extensive investigation by Rauszer (1980)
- Rauszer's completeness proofs fixed for the propositional case by Goré and Shillito (2020)
- Goré and Shillito's results mechanised in the Coq proof assistant by Shillito (2023)

Soundness and completeness are guaranteed to be correct!

Constructive Reverse Mathematics[1]

In foundations like constructive type theory, many sub-classical distinctions become visible:

$$\begin{aligned}
\text{Excluded Middle (LEM)} &:= \forall P : \mathbb{P}.\, P \lor \neg P \\
\text{Weak Excluded Middle (WLEM)} &:= \forall P : \mathbb{P}.\, \neg P \lor \neg\neg P \\
\text{Double Negation Shift (DNS)} &:= \forall p : \mathbb{N} \to \mathbb{P}.\, (\forall n.\, \neg\neg p\, n) \to \neg\neg(\forall n.\, p\, n) \\
\text{Markov's Principle (MP)} &:= \forall f : \mathbb{N} \to \mathbb{B}.\, \neg\neg(\exists n.\, f\, n = \mathrm{true}) \to \exists n.\, f\, n = \mathrm{true}
\end{aligned}$$

Some classically valid theorems are actually equivalent to constructively weaker principles...

Correct theorems can still be analysed regarding their logical strength!

[1] Ishihara (2006); Diener (2018)

Constructive Reverse Mathematics of Completeness Theorems

Does $\mathcal{T} \vDash \varphi$ imply $\mathcal{T} \vdash \varphi$ constructively?

Current situation in the literature on first-order logic:

- Completeness equivalent to Boolean Prime Ideal Theorem (Henkin, 1954)
- Completeness requires Markov's Principle (Kreisel, 1962)
- Completeness equivalent to Weak Kőnig's Lemma (Simpson, 2009)
- Completeness equivalent to Weak Fan Theorem (Krivtsov, 2015)
- Completeness holds fully constructively (Krivine, 1996)
- Systematic investigation is work in progress (Herbelin and Kirst, 2023)

Classical Outline for (Bi-)Intuitionistic Propositional Logic

Employing prime theories ($\varphi \vee \psi \in \mathcal{T} \to \varphi \in \mathcal{T} \vee \varphi \in \mathcal{T}$):

- Lindenbaum Extension: if $\mathcal{T} \nvdash \varphi$ then there is prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash \varphi$
- Universal Model $\mathcal{U}$: consistent prime theories related by inclusion
- Truth Lemma for $\mathcal{T}$ in $\mathcal{U}$: $\varphi \in \mathcal{T} \;\leftrightarrow\; \mathcal{T} \Vdash \varphi$
- Model Existence: if $\mathcal{T} \nvdash \varphi$ then there is $\mathcal{M}$ with $\mathcal{M} \Vdash \mathcal{T}$ and $\mathcal{M} \nVdash \varphi$
- Quasi-Completeness: if $\mathcal{T} \Vdash \varphi$ then $\neg\neg(\mathcal{T} \vdash \varphi)$
- Completeness: if $\mathcal{T} \Vdash \varphi$ then $\mathcal{T} \vdash \varphi$

Constructive Completeness Proof???

For $\mathcal{T}$ quasi-prime ($\varphi \vee \psi \in \mathcal{T} \to \neg\neg(\varphi \in \mathcal{T} \vee \varphi \in \mathcal{T})$):

- Lindenbaum Extension: if $\mathcal{T} \not\vdash \varphi$ then there is quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \not\vdash \varphi$
- Universal Model: consistent quasi-prime theories related by inclusion
- Truth Lemma: fails immediately
- Model Existence: fails
- Quasi-Completeness: fails
- Completeness: needs MP/LEM depending on theory complexity and syntax fragment

Constructive Completeness Proof?

For $\mathcal{T}$ quasi-prime $(\varphi \vee \psi \in \mathcal{T} \to \neg\neg(\varphi \in \mathcal{T} \vee \varphi \in \mathcal{T}))$ and stable $(\neg\neg(\varphi \in \mathcal{T}) \to \varphi \in \mathcal{T})$:

- Lindenbaum Extension: if $\mathcal{T} \not\vdash \varphi$ then there is stable quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \not\vdash \varphi$
- Universal Model: consistent stable quasi-prime theories related by inclusion
- Truth Lemma: fails for disjunction
- Model Existence: fails
- Quasi-Completeness: fails
- Completeness: needs MP/LEM depending on theory complexity and syntax fragment

The Issue with Disjunction

Truth Lemma case for disjunctions $\varphi \vee \psi$:

$$
\begin{aligned}
\varphi \vee \psi \in \mathcal{T} &\overset{?}{\Leftrightarrow} \mathcal{T} \Vdash \varphi \vee \psi \\
&\overset{def}{\Leftrightarrow} \mathcal{T} \Vdash \varphi \;\vee\; \mathcal{T} \Vdash \psi \\
&\overset{IH}{\Leftrightarrow} \varphi \in \mathcal{T} \;\vee\; \psi \in \mathcal{T}
\end{aligned}
$$

- So we really need prime theories to interpret disjunctions
- Primeness from Lindenbaum Extension is constructive no-go

Model Existence via WLEM

Weak law of excluded middle WLEM $:= \forall P : \mathbb{P}. \neg P \vee \neg\neg P$

Lemma

Assuming WLEM, every stable quasi-prime theory is prime.

Proof.

Assume $\varphi \vee \psi \in \mathcal{T}$. Using WLEM, decide whether $\neg(\varphi \in \mathcal{T})$ or $\neg\neg(\varphi \in \mathcal{T})$. In the latter case, conclude $\varphi \in \mathcal{T}$ directly by stability. In the former case, derive $\psi \in \mathcal{T}$ using stability, since assuming $\neg(\psi \in \mathcal{T})$ on top of $\neg(\varphi \in \mathcal{T})$ contradicts quasi-primeness for $\varphi \vee \psi \in \mathcal{T}$. □

Classical proof outline works again up to Model Existence and Quasi-Completeness!

Backwards Analysis

Which logical principles are really necessary for the intermediate statements?

Fact

Model Existence implies WLEM.

Proof.

Given P, use model existence on $\mathcal{T} := \{x_0 \lor \neg x_0\} \cup \{x_0 \mid P\} \cup \{\neg x_0 \mid \neg P\}$. We have $\mathcal{T} \nvdash \bot$ so if $\mathcal{M} \Vdash \mathcal{T}$, then either $\mathcal{M} \Vdash x_0$ or $\mathcal{M} \Vdash \neg x_0$, so either $\neg\neg P$ or $\neg P$, respectively. □

Fact

Quasi-Completeness implies the following principle: $\forall p : \mathbb{N} \to \mathbb{P}.\, \neg\neg(\forall n.\, \neg p\, n \lor \neg\neg p\, n)$

Proof.

Using similar tricks for $\mathcal{T} := \{x_n \lor \neg x_n\} \cup \{x_n \mid p\, n\} \cup \{\neg x_n \mid \neg p\, n\}$. □

Since Quasi-Completeness also follows from DNS, there is no hope it is equivalent to WLEM...

Weak Excluded-Middle Shift[2]

$$\text{WLEMS} := \forall p : \mathbb{N} \to \mathbb{P}. (\forall n. \neg\neg(\neg p\, n \vee \neg\neg p\, n)) \to \neg\neg(\forall n. \neg p\, n \vee \neg\neg p\, n)$$
$$\Leftrightarrow \forall pq : \mathbb{N} \to \mathbb{P}. (\forall n. \neg\neg(\neg p\, n \vee \neg q\, n)) \to \neg\neg(\forall n. \neg p\, n \vee \neg q\, n)$$

Lemma

Assuming WLEMS, every stable quasi-prime theory is not not prime.

Proof.

Assume $\mathcal{T}$ not prime and derive a contradiction. Given the negative goal, from WLEMS we obtain $\forall \varphi. \neg(\varphi \in \mathcal{T}) \vee \neg\neg(\varphi \in \mathcal{T})$. This yields exactly the instances of WLEM needed to derive that $\mathcal{T}$ is prime, contradiction. □

Already this lemma turns out to be enough for Quasi-Completeness!

[2] Mentioned in systematic study by Umezawa (1959) but absent from the literature otherwise

Quasi-Completeness via WLEMS

Refined proof outline using WLEMS:

- Lindenbaum Extension: if $\mathcal{T} \nvdash \varphi$ then there is stable not not prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash \varphi$
- Universal Model $\mathcal{U}$: consistent stable prime theories related by inclusion
- Truth Lemma for $\mathcal{T}$ in $\mathcal{U}$: $\varphi \in \mathcal{T} \ \leftrightarrow \ \mathcal{T} \Vdash \varphi$
- Quasi Model Existence: if $\mathcal{T} \nvdash \varphi$ then there not not is $\mathcal{M}$ with $\mathcal{M} \Vdash \mathcal{T}$ and $\mathcal{M} \nVdash \varphi$
- Quasi-Completeness: if $\mathcal{T} \Vdash \varphi$ then $\neg\neg(\mathcal{T} \vdash \varphi)$
- Completeness: needs MP/LEM depending on theory complexity and syntax fragment

Consequences and Open Questions

Consequences:

- WLEM and Model Existence are equivalent
- WLEMS, Quasi Model Existence, and Quasi-Completeness are equivalent
- WLEMS and MP together imply completeness for enumerable contexts

Open questions:

- What restriction of WLEMS is sufficient for enumerable contexts?
- What is the relation of WLEMS to the fan theorem?
- What is the constructive status of the traditional semantics of bi-intuitionistic logic?
- What observations transport to first-order bi-intuitionistic logic (or other logics)?

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Working Towards an Explanation

There are multiple dimensions at play:

- Syntax fragment (e.g., propositional, minimal, negative, full)
- Complexity of the context (e.g., finite, decidable, enumerable, arbitrary)
- Cardinality of the signature (e.g., countable, uncountable)
- Representation of the semantics (e.g., Boolean, decidable, propositional)

Ongoing systematic investigation using Coq:

- Started by Herbelin and Ilik (2016) and Forster, Kirst, and Wehr (2021)
- New observations by Hagemeier and Kirst (2022) and Kirst (2022)
- Comprehensive overview of current landscape by Herbelin (2022)
- Today: syntactic disjunction, arbitrary contexts, countable signature, prop. semantics

The Case of Soundness

Some axioms of like $\psi \vee (\top \prec \psi)$ are only valid in models behaving classically:

$$\text{decidable models} \subseteq \text{stable models} \subseteq \text{axiomatic models}$$

Lemma

Soundness holds for every model satisfying the (critical) axioms.

Corollary

Assuming LEM, soundness holds for all models.

Fact

Soundness for all models implies LEM.

Proof.

For a proposition P consider the single-world model with $w \Vdash x_0$ iff P. By assuming soundness, we have $w \Vdash x_0 \vee (\top \prec x_0)$ which is equivalent to $P \vee \neg P$. □

Quasi-Completeness via DNS

Assuming double-negation shift DNS $:= \forall X. \forall p : X \to \mathbb{P}. (\forall x. \neg\neg p\,x) \to \neg\neg(\forall x. p\,x)$:

- Lindenbaum Extension: if $\mathcal{T} \nvdash \varphi$ then there is stable quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash \varphi$
- Universal Model $\mathcal{U}$: consistent stable quasi-prime theories related by inclusion
- Pseudo Truth Lemma for $\mathcal{T}$ in $\mathcal{U}$: $\varphi \in \mathcal{T} \;\leftrightarrow\; \neg\neg(\mathcal{T} \Vdash \varphi)$
- Pseudo Model Existence: if $\mathcal{T} \nvdash \varphi$ then there is $\mathcal{M}$ with $\neg\neg(\mathcal{M} \Vdash \mathcal{T})$ and $\mathcal{M} \nVdash \varphi$
- Quasi-Completeness: if $\mathcal{T} \Vdash \varphi$ then $\neg\neg(\mathcal{T} \vdash \varphi)$ (also since DNS $\leftrightarrow \neg\neg$LEM)
- Completeness: needs MP/LEM depending on theory complexity and syntax fragment