The Generalised Continuum Hypothesis Implies the Axiom of Choice in Coq

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Sierpiński’s Theorem*

**Generalised Continuum Hypothesis (GCH):**
There are no cardinalities between an infinite set and its power set.

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5. Use GCH to iteratively squeeze in $\aleph(X)$ and obtain $|X| \leq |\aleph(X)|$

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Goal: Mechanisation in Coq
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Proof outline surprisingly abstract, only need to find formal notions of:

- Power sets
- Numbers
- Relations
- Functions
- Cardinality
- Orderings

An expressive type theory like Coq's type theory allows two strategies:

1. Axiomatise some variant of set theory
2. Use Coq itself to represent the necessary notions

Why are both variants interesting?

1. Many renderings of axiomatic set theory in type theory
2. Insights about type theory itself
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Variant 1: First-Order vs. Higher-Order ZF

Common setting: work in model $S : T$ providing set-theoretic structure

$$\in : S \to S \to \mathcal{P} \quad \cup : S \to S \quad \emptyset : S$$

$$\{\_, \_\} : S \to S \to S \quad \mathcal{P} : S \to S \quad \omega : S$$
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First-order ZF adds replacement for first-order relations:

$$\{x \mid \exists z \in y. \varphi(z, x)\} \quad (\varphi \text{ a functional first-order formula})$$
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- Streamlined infinity and foundation axioms (Kirst and Smolka (2018))
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Some abstract set-theoretic results apply to dependent type theories, e.g. the equivalence of WO and AC (cf. Ilik (2006); Smolka et al. (2015))
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\[
\forall X Y : \mathbb{T}. \forall (R : X \rightarrow Y \rightarrow \mathbb{P}). (\forall x. \exists y. R \, x \, y) \rightarrow \exists (f : X \rightarrow Y). \forall x. R \, x \, (f \, x)
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Three Levels of Set Theory in Coq

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Summary of our Paper

Sierpiński’s theorem already mechanised in Metamath by Carneiro (2015) based on a library of first-order ZF, we synthesise 3 alternatives in Coq:

- Coq* mechanisation based on higher-order ZF (2700loc)
- Adaptation to Coq* itself assuming unique choice (1400loc)
- Variant without unique choice (300loc on top)

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- Helpful features (type classes, setoid rewriting, auto rewriting)

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https://www.ps.uni-saarland.de/extras/sierpinski

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First Half in Higher-Order ZF
Higher-Order ZF Set Theory

Work in a model \((S, \in, \{\_ , \_\}, \cup, \mathcal{P}, \emptyset, \omega)\).
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Replace three of the usual first-order axioms by stronger versions:
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\forall A. \text{WF} \in A \quad \text{(Foundation)}
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\forall x. x \in \omega \iff \exists n : \mathbb{N}. x = \sigma^n(\emptyset) \quad \text{(Infinity)}
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\lambda y. \exists x \in A. R \times y \quad \text{is a set for all functional } R \quad \text{(Replacement)}
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Higher-order replacement yields a unique choice operator:

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\delta_p : \forall p : S \rightarrow \mathcal{P} \quad \text{if } \exists! A. pA \quad \Rightarrow \quad \sigma A. pA = \bigcup \{y \mid \exists x \in \mathcal{P}(\emptyset). py\}
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\delta p := \bigcup \{y | \exists x \in \mathcal{P}(\emptyset). py\}
\]

Collapses total functional relations and functions on \(S\) as expected!
Inductive Ordinals*

Definition

A set $x$ is transitive if every element is a subset ($z \in y \in x \rightarrow z \in x$).

*Gert Smolka (2016); Smullyan and Fitting (2010)
Inductive Ordinals

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A set $x$ is **transitive** if every element is a subset ($z \in y \in x \rightarrow z \in x$).

The class $\mathcal{O} : S \rightarrow \mathcal{P}$ of **ordinals** can be defined inductively by a single rule:

\[
\frac{\alpha \subseteq \mathcal{O} \text{ transitive } \alpha}{\alpha \in \mathcal{O}}
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---

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$$
\begin{align*}
\alpha \subseteq \mathcal{O} & \quad \text{transitive } \alpha \\
\alpha & \in \mathcal{O}
\end{align*}
$$

Equivalently, one can characterise $\mathcal{O}$ with 3 rules unveiling constructors:

$$
\begin{align*}
\emptyset & \in \mathcal{O} \\
\alpha & \in \mathcal{O} \quad \sigma(\alpha) \in \mathcal{O} \\
\lambda & \subseteq \mathcal{O} \quad (\bigcup \lambda \subseteq \lambda) \\
\bigcup \lambda & \in \mathcal{O}
\end{align*}
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Equivalently, one can characterise $\mathcal{O}$ with 3 rules unveiling constructors:

$$\frac{}{\emptyset \in \mathcal{O}} \quad \frac{\alpha \in \mathcal{O}}{\sigma(\alpha) \in \mathcal{O}} \quad \frac{\lambda \subseteq \mathcal{O} \quad (\bigcup \lambda \subseteq \lambda)}{\bigcup \lambda \in \mathcal{O}}$$

By simple induction on $\mathcal{O}$, one obtains the desired ordering properties:

Fact

Every ordinal is well-ordered by $\in$ and order-isomorphic ordinals are equal.

*Gert Smolka (2016); Smullyan and Fitting (2010)
Constructing Large Ordinals: $|ℕ(A)| ≠ |A|$

**Definition**

The Hartogs number of a set $A$ is the class $ℕ(A) := \lambda \alpha \in \mathcal{O}. |\alpha| ≤ |A|$. 
Constructing Large Ordinals: $|\aleph(A)| \nleq |A|$ 

**Definition**

The **Hartogs number** of a set $A$ is the class $\aleph(A) := \lambda \alpha \in \mathcal{O}. |\alpha| \leq |A|$.

**Theorem**

The Hartogs number $\aleph(A)$ of $A$ satisfies the following properties:

1. $|\aleph(A)| \leq |\mathcal{P}^6(A)|$
2. $\aleph(A) \in \mathcal{O}$
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Constructing Large Ordinals: $|\aleph(A)| \not\leq |A|$

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**Proof.**

1. By representing ordinals $|\alpha| \leq |A|$ as well-ordered subsets of $A$.
2. Straightforward by definition of ordinals.
3. Straightforward by definition of $\aleph(A)$.
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Second Half in Coq’s Type Theory
Small Ordinals in Type Theory

How to construct Hartogs numbers in Coq’s type theory?
No canonical representation of well-orders as ordinals*

*without quotient axioms or univalence
Small Ordinals in Type Theory

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Consider **small ordinals** representable in a given type $X$:

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D. Kirst and F. Rech

Sierpiński’s Theorem in Coq

CPP’21, January 17-19
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- $H(X)$ is defined as the subtype of small ordinals $\alpha$

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- $H(X)$ is defined as the subtype of small ordinals $\alpha$

Theorem

$H(X)$ is well-ordered and satisfies $|H(X)| \not\leq |X|$ and $|H(X)| \leq |\mathcal{P}^3(X)|$.

*without quotient axioms or univalence

---

D. Kirst and F. Rech

Sierpiński’s Theorem in Coq

CPP’21, January 17-19
Sierpiński’s Theorem - Proof

Theorem

$GCH$ implies $AC$. 

Proof.
Assume $GCH$, it suffices to show that every infinite type is well-orderable.

So for some infinite $X$, apply $GCH$ to the situation obtained by Lemma 1:

$|P_2(X)| \leq |P_2(X)| + |H(X)| \leq |P_3(X)|$

yields $|H(X)| \leq |P_2(X)|$, start again

Lemma 1
If $X$ is infinite, then $|X| = |1 + X|$ and $|P(X)| = |P(X) + P(X)|$. 

D. Kirst and F. Rech
Sierpiński’s Theorem in Coq
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### Sierpiński’s Theorem - Proof

**Theorem**

*GCH implies AC.*

**Proof.**

Assume GCH, it suffices to show that every infinite type is well-orderable. So for some infinite $X$, apply GCH to the situation obtained by Lemma 1:

$$|\mathcal{P}^2(X)| \leq |\mathcal{P}^2(X) + H(X)| \leq |\mathcal{P}^3(X)|$$

**Lemma 1**

*If $X$ is infinite, then $|X| = |\mathbb{1} + X|$ and $|\mathcal{P}(X)| = |\mathcal{P}(X) + \mathcal{P}(X)|$.***
Sierpiński’s Theorem - Proof

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GCH implies AC.

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\[ |P^2(X)| \leq |P^2(X) + H(X)| \leq |P^3(X)| \]

- $|P^2(X) + H(X)| \leq |P^2(X)|$ yields $|H(X)| \leq |P^2(X)|$, start again
- $|P^3(X)| \leq |P^2(X) + H(X)|$ yields $|P^3(X)| \leq |H(X)|$ by Lemma 2

Lemma 2

If $|P(X)| \leq |X + Y|$ and $|X + X| \leq |X|$, then already $|P(X)| \leq |Y|$.
Infinite Types: $|\mathcal{P}(X)| = |\mathcal{P}(X) + \mathcal{P}(X)|$
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Given types $X$, $Y$, a predicate $p : X \rightarrow \mathbb{P}$, and an injection $f : X \rightarrow Y$:

$|\mathbb{N}| = |1 + \mathbb{N}|$

$|\mathbb{B}| \overset{\text{UC}}{=} |\mathcal{P}|$

$|X + X| = |\mathbb{B} \times X|$

$|X| \overset{\text{UC}}{=} |\Sigma x. px + \Sigma x. \neg px|$

$|\mathcal{P}(X + Y)| = |\mathcal{P}(X) \times \mathcal{P}(Y)|$

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Given types \(X, Y\), a predicate \(p : X \to \mathbb{P}\), and an injection \(f : X \to Y\):

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\begin{align*}
|\mathbb{N}| &= |1 + \mathbb{N}| \\
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|X + X| &= |\mathbb{B} \times X| \\
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|\mathcal{P}(X + Y)| &\overset{UC}{=} |\mathcal{P}(X) \times \mathcal{P}(Y)| \\
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Lemma 1

*If \(X\) is infinite, then \(|X| \overset{UC}{=} |1 + X|\) and \(|\mathcal{P}(X)| \overset{UC}{=} |\mathcal{P}(X) + \mathcal{P}(X)|\).*
Infinite Types: $|\mathcal{P}(X)| = |\mathcal{P}(X) + \mathcal{P}(X)|$

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**Proof.**

By equational reasoning, e.g. the former implies the latter as follows:
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*If* $X$ *is infinite, then* $|X| \overset{\text{UC}}{=} |1 + X|$ *and* $|\mathcal{P}(X)| \overset{\text{UC}}{=} |\mathcal{P}(X) + \mathcal{P}(X)|$.

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Eliminating Unique Choice

1. Introduce weaker notions \(|X| \leq_r |Y|\) and \(|X| =_r |Y|\) based on injective and invertible total functional relations instead of functions.
Eliminating Unique Choice

1. Introduce weaker notions $|X| \leq_r |Y|$ and $|X| =_r |Y|$ based on injective and invertible total functional relations instead of functions.

2. Obtain the critical relational bijection without UC:
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\forall XY. |\mathbb{N}| \leq |X| \leq_r |Y| \leq_r |\mathcal{P}(X)| \rightarrow |Y| \leq_r |X| \lor |\mathcal{P}(X)| \leq_r |Y|
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$$\forall X,Y. \forall R: X \to Y \to \mathbb{P}. (\forall x. \exists y. Rxy) \to \exists R' \subseteq R. \forall x. \exists! y. R'xy$$

**Theorem**

$\text{GCH}'$ implies $\text{AC}'$. 

D. Kirst and F. Rech

Sierpiński’s Theorem in Coq

CPP’21, January 17-19
Wrap-Up
Take-Homes

Three ways to mechanise set-theoretic results in type-theoretic systems:

- **First-order axiomatisation** unavoidable for meta-theoretic results
- **Higher-order axiomatisation** available for internal results
- **Type-level structure** sometimes sufficient for abstract results

In this setting, higher-order ZF is a bridge between both worlds:

- Explicit set-theoretic primitives and notions
- Inheritance of type-theoretic structure
- Convenient to work with, especially without library support
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Open Questions

- How constructive is the main GCH to AC implication?
  - Mostly needed for ordering properties (linearity, WF)
  - Maybe factoring through the classical WO not necessary
  - Would show that GCH implies excluded middle

What is the situation in other type theories?

- MLTT: lacks a direct notion of propositional existence and power sets
- Type theory with AC: renders Sierpiński's theorem vacuous
- HoTT: probably a good target since FE, PE, and UC are provable

How connected are GCH on type-level and in the set-level model?

- Certainly the former implies the latter
- Converse implication probably independent
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