

Undecidability, Incompleteness, and Completeness of Second-Order Logic in Coq

Certified Programs and Proofs

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First-Order Logic

Quantification only over individuals

Second-Order Logic

Quantification over individuals & their properties

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Results well known (e.g. [Shapiro, 1991]). We analyse them in constructive type theory and mechanise them using the Coq proof assistant.

Given a signature $\Sigma = (\Sigma_{\mathcal{F}}, \Sigma_{\mathcal{P}})$, we inductively define

$$t ::= x_i \mid \mathcal{F} \vec{t} \quad (\mathcal{F} : \Sigma_{\mathcal{F}}) (i : \mathbb{N})$$

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- Unique challenges of SOL: arities, function quantifiers

Full Semantics: Undecidability and Incompleteness

Definition (Full Semantics)

- A model \mathcal{M} consists of a domain D and interpretations $\mathcal{F}^{\mathcal{M}} : D^{|\mathcal{F}|} \rightarrow D$ and $\mathcal{P}^{\mathcal{M}} : D^{|\mathcal{P}|} \rightarrow \text{Prop}$.

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- Interpretation (\models) in \mathcal{M} maps connectives \square and quantifiers ∇ to their counterparts in Prop .
- SOL quantifiers ∇_2^n range over the full relation space $D^n \rightarrow \text{Prop}$.

Second-Order Peano Arithmetic

Zero Addition : $\forall x. 0 + x \equiv x$

Addition Recursion : $\forall xy. (Sx) + y \equiv S(x + y)$

Disjointness : $\forall x. 0 \equiv Sx \rightarrow \perp$

Equality Reflexivity : $\forall x. x \equiv x$

Zero Multiplication : $\forall x. 0 \cdot x \equiv 0$

Multiplication Recursion : $\forall xy. (Sx) \cdot y \equiv y + x \cdot y$

Successor Injectivity : $\forall xy. Sx \equiv Sy \rightarrow x \equiv y$

Equality Symmetry : $\forall xy. x \equiv y \rightarrow y \equiv x$

Induction : $\forall P. P(0) \rightarrow (\forall x. P(x) \rightarrow P(Sx)) \rightarrow \forall x. P(x)$

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PA_2 is categorical for full semantics, i.e. all models of PA_2 are isomorphic.

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Proof.

Given models $\mathcal{M}_1, \mathcal{M}_2 \models PA_2$, inductively define $\cong : D_1 \rightarrow D_2 \rightarrow \text{Prop}$

$$0^{\mathcal{M}_1} \cong 0^{\mathcal{M}_2} \quad S^{\mathcal{M}_1} x \cong S^{\mathcal{M}_2} y \quad \text{if } x \cong y.$$

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Verify that \cong is an isomorphism using the induction axiom. □

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- But \mathbb{N} is not model of the whole theory \mathcal{T}_{\neq} . Since \mathbb{N} is the only model of PA_2 , we can conclude that \mathcal{T}_{\neq} does not have a model. \square

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- P not enumerable if a not enumerable problem reduces to it (e.g. $\overline{\text{Halt}}$).

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Proof Sketch.

Via reduction from the complement of Hilbert's tenth problem $\overline{H_{10}}$:¹

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- $s = t$ has a solution iff $\exists O S + \times \equiv. PA_2 \wedge \varphi_{s,t}$ is satisfiable. □

Henkin Semantics: Completeness

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- Second-order quantifiers ∇_2^n only range over the relations contained in a universe $\mathbb{U}_n : (D^n \rightarrow \text{Prop}) \rightarrow \text{Prop}$.
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The second-order ND system \vdash_2 is obtained by extending the first-order system \vdash_1 with rules for second-order quantifiers and comprehension:

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\vdash_2 is complete for Henkin semantics [Henkin, 1949].

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x and p represent individuals and predicates at the same time.

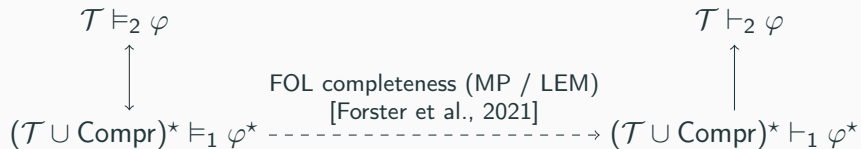
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$$\begin{array}{c} \mathcal{T} \models_2 \varphi \\ \updownarrow \\ (\mathcal{T} \cup \text{Compr})^* \models_1 \varphi^* \end{array}$$

Reduction to FOL [Nour and Raffalli, 2003]

$$\begin{array}{ccc} \mathcal{T} \models_2 \varphi & & \\ \updownarrow & \text{FOL completeness (MP / LEM)} & \\ (\mathcal{T} \cup \text{Compr})^* \models_1 \varphi^* & \text{[Forster et al., 2021]} & \dashrightarrow (\mathcal{T} \cup \text{Compr})^* \vdash_1 \varphi^* \end{array}$$

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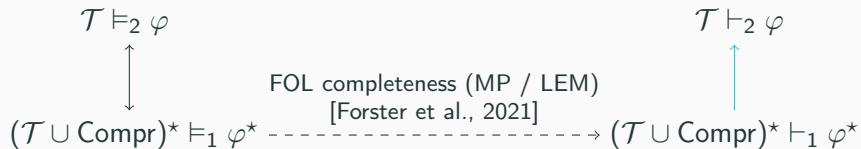


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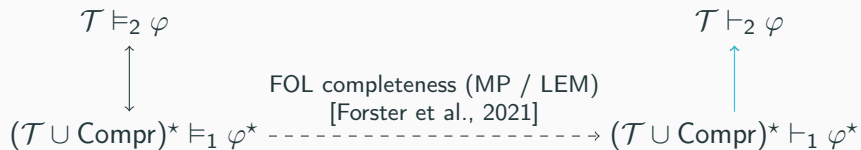
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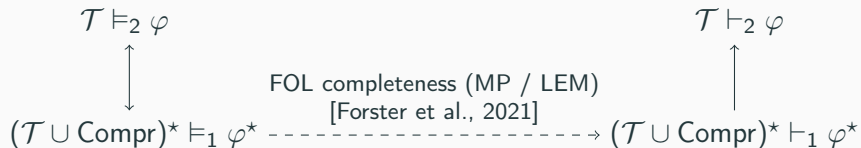
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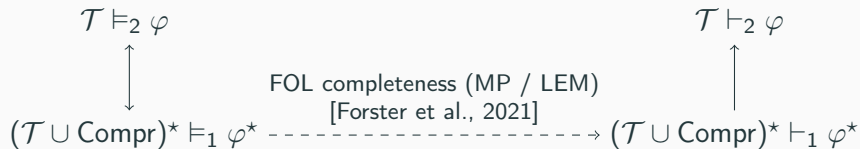


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Theorem (Relative Completeness)

If FOL is complete, then so is SOL with Henkin semantics.



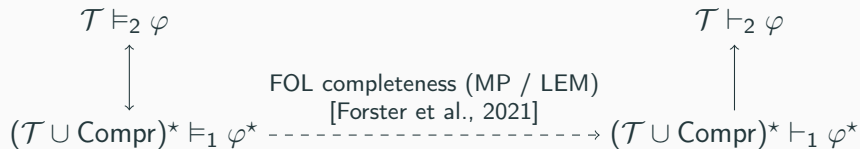
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SOL with Henkin semantics is complete under LEM.



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Theorem (Completeness)

SOL with Henkin semantics is complete under LEM.

Theorem (Compactness)

SOL with Henkin semantics is compact under LEM.

Theorem (Relative Löwenheim-Skolem)

If FOL has the Löwenheim-Skolem property, then so does SOL with Henkin semantics.

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- Other second-order axiomatisations, e.g. ZF_2
- Internal Categoricity [Väänänen and Wang, 2012]. Would require extensive tooling, maybe similar to the proof mode in [Hostert et al., 2021].



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Theorem (Failure of Strong Completeness).

SOL is not strongly complete for full semantics and decidable theories.

Proof.

Let \vdash be sound and strongly complete.

There is no model of \mathcal{T}_{\neq} . Thus

$$\mathcal{T}_{\neq} \models \perp \xrightarrow{\text{Completeness}} \mathcal{T}_{\neq} \vdash \perp \rightarrow \text{for } \Gamma \subseteq_{\text{fin}} \mathcal{T}_{\neq} \quad \Gamma \vdash \perp \xrightarrow{\text{Soundness}} \Gamma \models \perp$$

But $\Gamma \subseteq_{\text{fin}} \mathcal{T}_{\neq}$ has a model. □

Every function definable in constructive type theory is computable.

This allows a synthetic rendering of computability theory without relying on a concrete model of computation.

A problem $P : X \rightarrow \text{Prop}$...

- is **decidable** if $\exists f : X \rightarrow \mathbb{B}. \forall x. P(x) \leftrightarrow f(x) = \text{true}$.
- is **enumerable** if $\exists f : \mathbb{N} \rightarrow \mathcal{O}(X). \forall x. P(x) \leftrightarrow \exists n. f(n) = x$.
- **reduces to** $Q : Y \rightarrow \text{Prop}$ if $\exists f : X \rightarrow Y. \forall x. P(x) \leftrightarrow Q(f(x))$.

Semantic Henkin Reduction

- Turn Henkin model \mathcal{H} into first-order model \mathcal{H}^* with $D^* := D \cup \mathbb{U}$ and $\text{App}_n(x :: \vec{v}) := \text{toPred}_n x (\text{toIndi } \vec{v})$

$$\mathcal{H} \models_2 \varphi \leftrightarrow \mathcal{H}^* \models_1 \varphi^*$$

- Turn first-order model \mathcal{M} into Henkin model \mathcal{M}^\diamond with $D^\diamond := D$ and \mathbb{U} induced by interpretation of App.

$$\mathcal{M} \models_1 \text{Compr}^* \rightarrow \mathcal{M}^\diamond \models_2 \varphi \leftrightarrow \mathcal{M} \models_1 \varphi^*$$

Backwards Translation

Define a backwards translation $_^\diamond : \text{form}_1 \rightarrow \text{form}_2$. For example

$$(\forall x. \text{App}_0(x) \dot{\wedge} \text{App}_1(x, x))^\diamond$$

$$\parallel$$

$$\forall x X^0 X^1. X^0 \dot{\wedge} X^1(x)$$

$$(\text{App}_1(f(x), y))^\diamond = \dot{\perp}_1(y)$$

Special error symbols $\dot{\perp}_n$ if first argument is not a variable

Consider a theory \mathcal{T} depending on a single predicate symbol \mathcal{P}

$$\text{Categ}(\mathcal{T}) := \forall D_1 D_2 P_1 P_2. \mathcal{T}(P_1)^{D_1} \dashv\vdash \mathcal{T}(P_2)^{D_2} \dashv\vdash \exists \cong . \text{Iso}(\cong, D_1, D_2, P_1, P_2)$$

where $\mathcal{T}(P_1)^{D_1}$ replaces \mathcal{P} with the variable P_1 and guards all quantifiers with the domain predicate D_1 .

- \mathcal{T} is categorical iff $\models \text{Categ}(\mathcal{T})$
- Provable in many cases (despite incompleteness), e.g. $\vdash \text{Categ}(\text{PA}_2)$.
 \Rightarrow Categoricity can be written and proven at the object level, without depending on any external set theory (or type theory in our case)