A Mechanised and Constructive Reverse Analysis of Soundness and Completeness of Bi-Intuitionistic Logic

Ian Shillito and Dominik Kirst

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Bi-Intuitionistic Logic

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\[ w \models \varphi \rightarrow \psi := \forall w'. w' \models \varphi \rightarrow w' \models \psi \]
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\[ w \vDash \varphi \leftarrow \psi \ := \ \neg (\forall w' \leq w. w' \vDash \varphi \rightarrow w' \vDash \psi) \]
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\[ w \models \varphi \leftarrow \neg \psi := \exists w' \leq w. \quad w' \models \varphi \land w' \not\models \psi \]
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Extends intuitionistic logic with exclusion, a dual to implication:

\[
\begin{align*}
    w \triangleright \varphi \rightarrow \psi & := \forall w' \geq w. w' \triangleright \varphi \rightarrow w' \triangleright \psi \\
    w \triangleright \varphi \leftarrow \neg \psi & := \neg(\forall w' \leq w. w' \triangleright \varphi \rightarrow w' \triangleright \psi) \\
    w \triangleright \varphi \leftarrow \neg \psi & := \exists w' \leq w. w' \triangleright \varphi \wedge w' \not\triangleright \psi
\end{align*}
\]

Corresponds to extending proof calculi with axioms for exclusion, e.g.

\[
\psi \lor (\top \leftarrow \psi)
\]
capturing the case distinction \( w \triangleright \varphi \lor \neg(\forall w' \leq w. w' \triangleright \varphi) \).
A Case for Computer Mechanisation
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Soundness and completeness are guaranteed to be correct!
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Constructive Reverse Mathematics$^1$

In foundations like constructive type theory, many sub-classical distinctions become visible:

1. Excluded Middle (LEM): $\forall P : P \lor \neg P$
2. Weak Excluded Middle (WLEM): $\forall P : P \lor \neg \neg P$
3. Double Negation Shift (DNS): $\forall p : N \rightarrow P. (\forall n. \neg \neg p n) \rightarrow \neg \neg (\forall n. p n)$
4. Markov's Principle (MP): $\forall f : N \rightarrow B. \neg \neg (\exists n. f n = true) \rightarrow \exists n. f n = true$

Some classically valid theorems are actually equivalent to constructively weaker principles...

Correct theorems can still be analysed regarding their logical strength!

$^1$Ishihara (2006); Diener (2018)
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The Case of Soundness

Some axioms of Bi-Int like $\psi \lor (T \rightarrow \psi)$ are only valid in models behaving classically:

- decidable models $\subseteq$ stable models $\subseteq$ axiomatic models
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**Fact**

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**Proof.**

For a proposition $P$ consider the single-world model with $w \models x_0$ iff $P$. By assuming soundness, we have $w \models x_0 \lor (\top \rightarrow x_0)$ which is equivalent to $P \lor \neg P$. \qed
Constructive Reverse Mathematics of Completeness Theorems

Does $\mathcal{T} \vDash \varphi$ imply $\mathcal{T} \vdash \varphi$ constructively?
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Current situation in the literature on first-order logic:

- Completeness equivalent to Boolean Prime Ideal Theorem (Henkin, 1954)
- Completeness requires Markov’s Principle (Kreisel, 1962)
- Completeness equivalent to Weak König’s Lemma (Simpson, 2009)
- Completeness equivalent to Weak Fan Theorem (Krivtsov, 2015)
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- Completeness equivalent to Weak König’s Lemma (Simpson, 2009)
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- Completeness holds fully constructively (Krivine, 1996)
Working Towards an Explanation

There are multiple dimensions at play:

- Syntax fragment (e.g., propositional, minimal, negative, full)
- Complexity of the context (e.g., finite, decidable, enumerable, arbitrary)
- Cardinality of the signature (e.g., countable, uncountable)
- Representation of the semantics (e.g., Boolean, decidable, propositional)

Ongoing systematic investigation using Coq:

- Started by Herbelin and Ilik (2016), picked up by Forster, Kirst, and Wehr (2021)
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Today: syntactic disjunction, arbitrary contexts, countable signature, prop. semantics
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Classical Outline for (Bi-)Intuitionistic Propositional Logic

Employing prime theories \((\varphi \lor \psi \in \mathcal{T} \rightarrow \varphi \in \mathcal{T} \lor \varphi \in \mathcal{T})\):

- **Lindenbaum Extension**: if \(\mathcal{T} \not\vdash \varphi\) then there is prime \(\mathcal{T}' \supseteq \mathcal{T}\) with \(\mathcal{T}' \not\vdash \varphi\).

- **Universal Model** \(U\): consistent prime theories related by inclusion.

- **Truth Lemma** for \(\varphi \in \mathcal{T}\) in \(U\):
  \(\varphi \in \mathcal{T} \iff \mathcal{T} \models \varphi\).

- **Model Existence**: if \(\mathcal{T} \not\vdash \varphi\) then there is \(M\) with \(M \models \mathcal{T}\) and \(M \not\models \varphi\).

- **Quasi-Completeness**: if \(\mathcal{T} \models \varphi\) then \(\neg\neg(\mathcal{T} \vdash \varphi)\).

- **Completeness**: if \(\mathcal{T} \models \varphi\) then \(\mathcal{T} \vdash \varphi\).
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Classical Outline for (Bi-)Intuitionistic Propositional Logic

Employing prime theories \((\varphi \lor \psi \in T \rightarrow \varphi \in T \lor \varphi \in T)\):

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- Universal Model \(U\): consistent prime theories related by inclusion
- Truth Lemma for \(T\) in \(U\): \(\varphi \in T \iff T \models \varphi\)
- Model Existence: if \(T \not\vdash \varphi\) then there is \(M\) with \(M \models T\) and \(M \not\models \varphi\)
- Quasi-Completeness: if \(T \models \varphi\) then \(\neg \neg (T \vdash \varphi)\)
Classical Outline for (Bi-)Intuitionistic Propositional Logic

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Constructive Completeness Proof???

For $\mathcal{T}$ quasi-prime ($\varphi \lor \psi \in \mathcal{T} \rightarrow \neg \neg (\varphi \in \mathcal{T} \lor \varphi \in \mathcal{T})$):

- Lindenbaum Extension: if $\mathcal{T} \not\vdash \varphi$ then there is quasi-prime $\mathcal{T}' \supseteq \mathcal{T}$ with $\mathcal{T}' \not\vdash \varphi$

- Universal Model: consistent quasi-prime theories related by inclusion

- Truth Lemma: fails immediately

- Model Existence: fails

- Quasi-Completeness: fails

- Completeness: needs MP/LEM depending on theory complexity and syntax fragment
Constructive Completeness Proof?

For $\mathcal{T}$ quasi-prime ($\varphi \lor \psi \in \mathcal{T} \rightarrow \neg \neg (\varphi \in \mathcal{T} \lor \varphi \in \mathcal{T})$) and stable ($\neg \neg (\varphi \in \mathcal{T}) \rightarrow \varphi \in \mathcal{T}$):

- Lindenbaum Extension: if $\mathcal{T} \not\vdash \varphi$ then there is stable quasi-prime $\mathcal{T}' \supseteq \mathcal{T}$ with $\mathcal{T}' \not\vdash \varphi$

- Universal Model: consistent stable quasi-prime theories related by inclusion

- Truth Lemma: fails for disjunction

- Model Existence: fails

- Quasi-Completeness: fails

- Completeness: needs MP/LEM depending on theory complexity and syntax fragment
The Issue with Disjunction

Truth Lemma case for disjunctions $\varphi \lor \psi$:

$$\varphi \lor \psi \in T \iff T \vdash \varphi \lor \psi$$

So we really need prime theories to interpret disjunctions. Primeness from Lindenbaum Extension is constructive no-go.
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Truth Lemma case for disjunctions $\varphi \lor \psi$:

$$\varphi \lor \psi \in \mathcal{T} \iff \mathcal{T} \not\models \varphi \lor \psi$$

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Model Existence via WLEM

Weak law of excluded middle WLEM := ∀P : P. ¬P ∨ ¬¬P

**Lemma**

Assuming WLEM, every *stable quasi-prime* theory is *prime.*
Model Existence via WLEM

Weak law of excluded middle WLEM \[ := \forall P : \mathbb{P}. \neg \neg P \lor \neg P \]

**Lemma**

Assuming WLEM, every *stable* quasi-prime theory is prime.

**Proof.**

Assume \( \varphi \lor \psi \in T \). Using WLEM, decide whether \( \neg (\varphi \in T) \) or \( \neg \neg (\varphi \in T) \). In the latter case, conclude \( \varphi \in T \) directly by stability. In the former case, derive \( \psi \in T \) using stability, since assuming \( \neg (\psi \in T) \) on top of \( \neg (\varphi \in T) \) contradicts quasi-primeness for \( \varphi \lor \psi \in T \). \( \square \)
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Classical proof outline works again up to Model Existence and Quasi-Completeness!
Backwards Analysis

Which logical principles are really necessary for the intermediate statements?

Fact
Model Existence implies WLEM.
Proof.
Given $P$, use model existence on $T := \{x \lor \neg x\} \cup \{x \mid P\} \cup \{\neg x \mid \neg P\}$. We have $T \not\vdash \bot$ so if $M \Vdash T$, then either $M \Vdash x \lor \neg x$ or $M \Vdash \neg x \lor \neg x$, so either $\neg \neg P$ or $\neg P$, respectively.

Fact
Quasi-Completeness implies the following principle: $\forall p : N \rightarrow P \neg \neg (\forall n. \neg p n \lor \neg \neg p n)$.
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Using similar tricks for $T := \{x \lor \neg x\} \cup \{x \mid p n\} \cup \{\neg x \mid \neg p n\}$.

Since Quasi-Completeness also follows from DNS, there is no hope it is equivalent to WLEM...
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Proof.

Given $P$, use model existence on $\mathcal{T} := \{x_0 \lor \neg x_0\} \cup \{x_0 \mid P\} \cup \{\neg x_0 \mid \neg P\}$. We have $\not\vdash_{\mathcal{T}} \bot$ so if $\mathcal{M} \models \mathcal{T}$, then either $\mathcal{M} \models x_0$ or $\mathcal{M} \models \neg x_0$, so either $\neg \neg P$ or $\neg P$, respectively. □
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**Proof.**

Using similar tricks for $\mathcal{T} := \{x_n \lor \neg x_n\} \cup \{x_n \mid p n\} \cup \{\neg x_n \mid \neg p n\}$.

Since Quasi-Completeness also follows from DNS, there is no hope it is equivalent to WLEM...
Weak Excluded-Middle Shift\(^2\)

\[
\text{WLEMS} := \forall p : \mathbb{N} \rightarrow \mathbb{P}. \quad \neg \neg (\forall n. \neg p n \lor \neg \neg p n)
\]

\(^2\)Mentioned in systematic study by Umezawa (1959) but absent from the literature otherwise
Weak Excluded-Middle Shift$^2$

\[
\text{WLEMS} := \forall p : \mathbb{N} \to \mathbb{P}. (\forall n. \neg\neg (\neg p n \lor \neg\neg p n)) \to \neg(\forall n. \neg p n \lor \neg\neg p n)
\]

\[\text{Lemma}\]
Assuming WLEMS, every stable quasi-prime theory is not not prime.

\[\text{Proof.}\]
Assume $T$ not prime and derive a contradiction. Given the negative goal, from WLEMS we obtain $\forall \phi. \neg (\phi \in T) \lor \neg\neg (\phi \in T)$. This yields exactly the instances of WLEM needed to derive that $T$ is prime, contradiction.

Already this lemma turns out to be enough for Quasi-Completeness!

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Weak Excluded-Middle Shift²

\[ \text{WLEMS} \ := \ \forall p : \mathbb{N} \rightarrow \mathbb{P}. \ (\forall n. \neg \neg (\neg p n \lor \neg \neg p n)) \rightarrow \neg \neg (\forall n. \neg p n \lor \neg \neg p n) \]
\[ \iff \forall pq : \mathbb{N} \rightarrow \mathbb{P}. \ (\forall n. \neg \neg (\neg p n \lor \neg q n)) \rightarrow \neg \neg (\forall n. \neg p n \lor \neg q n) \]

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**Lemma**

*Assuming WLEMS, every stable quasi-prime theory is not not prime.*

---

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\text{WLEMS} \equiv \forall p : \mathbb{N} \rightarrow \mathbb{P}. (\forall n. \lnot (\lnot p n \lor \lnot p n)) \rightarrow \lnot (\forall n. \lnot p n \lor \lnot p n)
\]

\[
\equiv \forall pq : \mathbb{N} \rightarrow \mathbb{P}. (\forall n. \lnot (\lnot p n \lor \lnot q n)) \rightarrow \lnot (\forall n. \lnot p n \lor \lnot q n)
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**Lemma**

*Assuming WLEMS, every stable quasi-prime theory is not not prime.*

**Proof.**

Assume \(\mathcal{T}\) not prime and derive a contradiction. Given the negative goal, from WLEMS we obtain \(\forall \varphi. \lnot (\varphi \in \mathcal{T}) \lor \lnot (\varphi \in \mathcal{T})\). This yields exactly the instances of WLEM needed to derive that \(\mathcal{T}\) is prime, contradiction.

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**Lemma**

*Assuming WLEMS, every stable quasi-prime theory is not not prime.*

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Assume \( \mathcal{T} \) not prime and derive a contradiction. Given the negative goal, from WLEMS we obtain \( \forall \varphi. \neg(\varphi \in \mathcal{T}) \lor \neg(\varphi \in \mathcal{T}) \). This yields exactly the instances of WLEM needed to derive that \( \mathcal{T} \) is prime, contradiction. \( \square \)

Already this lemma turns out to be enough for Quasi-Completeness!

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Quasi-Completeness via WLEMS

Refined proof outline using WLEMS:

Lindenbaum Extension: if $T \not\vdash \phi$ then there is stable not not prime $T'$ $\supseteq T$ with $T' \not\vdash \phi$

Universal Model $U$: consistent stable prime theories related by inclusion

Truth Lemma for $T$ in $U$: $\phi \in T \iff T \vDash \phi$

Quasi Model Existence: if $T \not\vdash \phi$ then not not is $M$ with $M \vDash T$ and $M \not\vDash \phi$

Quasi-Completeness: if $T \vDash \phi$ then $\neg \neg (T \vdash \phi)$

Completeness: needs MP/LEM depending on theory complexity and syntax fragment
Quasi-Completeness via WLEMS

Refined proof outline using WLEMS:

- Lindenbaum Extension: if $\mathcal{T} \not\vdash \varphi$ then there is stable not not prime $\mathcal{T}' \supseteq \mathcal{T}$ with $\mathcal{T}' \not\vdash \varphi$
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Refined proof outline using WLEMS:

- **Lindenbaum Extension**: if $T \not\vdash \varphi$ then there is **stable not not prime** $T' \supseteq T$ with $T' \not\vdash \varphi$

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- **Truth Lemma for $\mathcal{T}$ in $\mathcal{U}$**: $\varphi \in \mathcal{T} \iff \mathcal{T} \models \varphi$
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\neg(\mathcal{T} \vDash \varphi)$
Quasi-Completeness via WLEMS

Refined proof outline using WLEMS:

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- **Universal Model $U$:** consistent stable prime theories related by inclusion

- **Truth Lemma for $T$ in $U$:** $\phi \in T \iff T \models \phi$

- **Quasi Model Existence:** if $T \not\vdash \phi$ then there not not is $M$ with $M \models T$ and $M \not\models \phi$

- **Quasi-Completeness:** if $T \models \phi$ then $\neg\neg(T \vdash \phi)$

- **Completeness:** needs MP/LEM depending on theory complexity and syntax fragment
Mechanisation and Open Questions

Mechanisation:
- Coq development (5k loc) guarantees correctness of error-prone proofs
- Keeps track of subtle sub-classical changes in definitions and theorems
- Underlying constructive type theory allows fine-grained analysis

Open questions:
- What restriction of WLEMS is sufficient for enumerable contexts?
- What is the relation of WLEMS to the fan theorem?
- What is the constructive status of the traditional semantics of bi-intuitionistic logic?
- What observations transport to first-order bi-intuitionistic logic (or other logics?)
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Bibliography I


Quasi-Completeness via DNS

Assuming double-negation shift $\text{DNS} := \forall X. \forall p : X \rightarrow \mathbb{P}. (\forall x. \neg\neg p x) \rightarrow \neg\neg(\forall x. p x)$:

- **Lindenbaum Extension**: if $\mathcal{T} \not\vdash \varphi$ then there is stable quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \not\vdash \varphi$
- **Universal Model $\mathcal{U}$**: consistent stable quasi-prime theories related by inclusion
- **Pseudo Truth Lemma for $\mathcal{T}$ in $\mathcal{U}$**: $\varphi \in \mathcal{T} \iff \neg\neg(\mathcal{T} \vDash \varphi)$
- **Pseudo Model Existence**: if $\mathcal{T} \not\vdash \varphi$ then there is $\mathcal{M}$ with $\neg\neg(\mathcal{M} \vDash \mathcal{T})$ and $\mathcal{M} \not\vDash \varphi$
- **Quasi-Completeness**: if $\mathcal{T} \vDash \varphi$ then $\neg\neg(\mathcal{T} \vdash \varphi)$ (also since DNS $\leftrightarrow \neg\neg\text{LEM}$)
- **Completeness**: needs MP/LEM depending on theory complexity and syntax fragment