The Kleene-Post and Post's Theorem in the Calculus of Inductive Constructions

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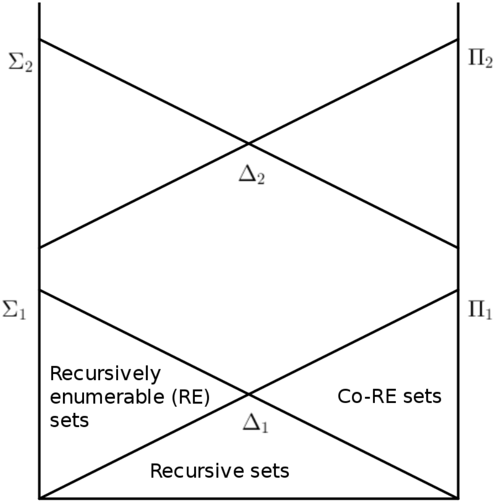
The Results: Kleene-Post and Post



https://en.wikipedia.org/wiki/Arithmetical_hierarchy

Theorem (Kleene-Post)

There are incomparable Turing degrees.

Theorem (Post)

1. *Turing degrees in Σ_{n+1} are semi-decidable relative to Turing degrees in Π_n.*
2. *The Turing degree of $\emptyset^{(n)}$ is many one-complete in Σ_n.*

The Method: Formal, Constructive, Synthetic

We work in a formal setting:

- Logical foundation is the Calculus of Inductive Constructions (CIC)
- All results are mechanised using the Coq proof assistant (hyperlinked with paper)

We use constructive logic:

- Refrain from classical reasoning where possible
- Analyse non-constructive assumptions where necessary

We adopt a synthetic perspective on computability:

- Leaves some invisible mathematics invisible
- Simplifies formal presentation of standard results

Synthetic Computability[1]

Exploit that in constructive foundations, every definable function is computable:

$$\text{a predicate } A : X \to \mathbb{P} \text{ is decidable} \quad := \quad \exists d : X \to \mathbb{B}.\, \forall x.\, A\,x \leftrightarrow d\,x = \text{true}$$

$$A : X \to \mathbb{P} \text{ many-one-reduces to } B : Y \to \mathbb{P} \quad := \quad \exists r : X \to Y.\, \forall x.\, A\,x \leftrightarrow B\,(r\,x)$$

Pros:

- Avoid manipulating Turing machines or equivalent model of computation
- Elegant formalisation (e.g. in CIC), feasible mechanisation (e.g. in Coq)

Cons:

- Finding a correct synthetic rendering of Turing reductions not so straightforward
- But Turing reductions are needed for interesting results like Kleene-Post and Post

[1] Richman (1983); Bauer (2006); Forster, Kirst and Smolka (2019)

Synthetic Oracle Computability

Some attempts: Bauer (2021); Forster (2021); Forster and Kirst (2022); Mück (2022)...

Definition (Forster, Kirst and Mück (2023))

An oracle computation is a functional $F : (Q \to A \to \mathbb{P}) \to I \to O \to \mathbb{P}$ captured by a computation tree $\tau : I \to A^* \rightharpoonup Q + O$ and its induced interrogation relation $\tau\, i\, ; R \vdash qs\, ; as$ as follows:

$$F\ R\ i\ o \ \leftrightarrow\ \exists qs\, as.\ \tau\, i\, ; R \vdash qs\, ; as \land \tau\, x\, as \triangleright \mathrm{out}\ o$$

$$A \preceq_T B \ := \ \text{there is an oracle computation } F : (\mathbb{N} \to \mathbb{B} \to \mathbb{P}) \to \mathbb{N} \to \mathbb{B} \to \mathbb{P} \text{ with } F\ B = A$$

$$\mathcal{S}_B(A) \ := \ \text{there is an oracle computation } F : (\mathbb{N} \to \mathbb{B} \to \mathbb{P}) \to \mathbb{N} \to \mathbb{1} \to \mathbb{P} \text{ with } \mathrm{dom}(F\ B) = A$$

Enumerating Oracle Computations

We need an enumeration of oracle computations for diagonalisations / Turing jump...

To ensure consistency, we start from a standard axiom (Kreisel (1970); Forster (2021)):

$$\mathsf{EPF} := \exists \theta : \mathbb{N} \to (\mathbb{N} \rightharpoonup \mathbb{N}).\, \forall f : \mathbb{N} \rightharpoonup \mathbb{N}.\, \exists c : \mathbb{N}.\, \forall x v.\, \theta_c\, x \triangleright v \leftrightarrow f\, x \triangleright v$$

Theorem

There is an enumerator of functionals $\chi : \mathbb{N} \to (\mathbb{N} \to \mathbb{B} \to \mathbb{P}) \to \mathbb{N} \to \mathbb{B} \to \mathbb{P}$ *such that*

1. χ_c *is an oracle computation for all* $c : \mathbb{N}$ *and*
2. *given an oracle computation* F *there exists* $c : \mathbb{N}$ *such that* $\forall R x b.\, \chi_c\, R\, x\, b \leftrightarrow F\, R\, x\, b$.

The Kleene-Post Theorem (à la Odifreddi (1992))

Goal: construct incomparable Turing degrees $A := \bigcup_{n:\mathbb{N}} \sigma_n$ and $B := \bigcup_{n:\mathbb{N}} \tau_n$

Characterise σ_n and τ_n inductively by a predicate $n \triangleright (\sigma, \tau)$ with:

- If $n \triangleright (\sigma, \tau)$ and $n \triangleright (\sigma', \tau')$, then $\sigma = \sigma'$ and $\tau = \tau'$
- For every n there not not exist σ and τ with $n \triangleright (\sigma, \tau)$
- If $2n \triangleright (\sigma, \tau)$, then $\chi_n\, A$ differs from B at position $|\tau|$
- If $2n + 1 \triangleright (\sigma, \tau)$, then $\chi_n\, B$ differs from A at position $|\sigma|$

Theorem (Kleene-Post)

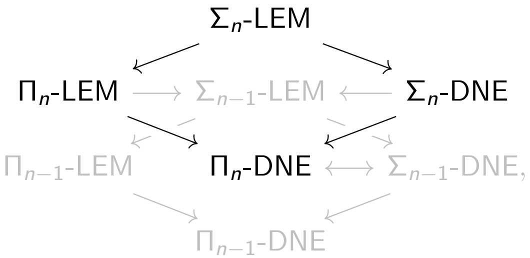
There are predicates A and B such that neither $A \preceq_T B$ nor $B \preceq_T A$.

The Arithmetical Hierarchy and Classical Axioms

Represent the arithmetical hierarchy on predicates $p : \mathbb{N}^k \to \mathbb{P}$ inductively:

$$\frac{f : \mathbb{N}^k \to \mathbb{B}}{\Sigma_0(\lambda \vec{x}.\, f\,\vec{x} = \mathsf{true})} \qquad \frac{f : \mathbb{N}^k \to \mathbb{B}}{\Pi_0(\lambda \vec{x}.\, f\,\vec{x} = \mathsf{true})} \qquad \frac{\Pi_n\, p}{\Sigma_{n+1}(\lambda \vec{x}.\, \exists y.\, p\,(y :: \vec{x}))} \qquad \frac{\Sigma_n\, p}{\Pi_{n+1}(\lambda \vec{x}.\, \forall y.\, p\,(y :: \vec{x}))}$$

With $\mathsf{LEM} := \forall P : \mathbb{P}.\, P \lor \neg P$ and $\mathsf{DNE} := \forall P : \mathbb{P}.\, \neg\neg P \to P$ we have (Akama et al. (2004)):



Post's Theorem (à la Odifreddi (1992))

Turing jump A' of A $:= \lambda n.\, \chi_n\, A\, n\, \mathsf{true}$

Satisfies $\mathcal{S}_A(A')$ while $\neg\mathcal{S}_A(\overline{A'})$, so iterated jumps $A^{(n)}$ yield a hierarchy of harder problems...

Theorem (Post)

Assuming Σ_n-LEM, the following can be shown:

- *Predicates are in Σ_{n+1} iff they are semi-decidable relative to predicates in Π_n.*
- *$\emptyset^{(n)}$ is in Σ_n and every other predicate in Σ_n many-one reduces to $\emptyset^{(n)}$.*

Proof.

- Direction $\rightarrow$ by linear search (Π_n-LEM), direction $\leftarrow$ by running tree (Σ_n-DNE).
- By the previous result and since $\mathcal{S}_B(A)$ iff A many-one reduces to B'. □

Outlook

1. Show that the incomparable degrees from Kleene-Post are below $\emptyset'$
   $\Rightarrow$ Extend the abstract interface describing oracle computations suitably

2. Analyse use of LEM, enabled by mechanisation, enabled by synthetic formalisation
   $\Rightarrow$ Avoid switching between Σ_n and Π_n via complementation (Akama et al. (2004))

3. Tackle Post's problem regarding an undecidable but enumerable degree below $\emptyset'$
   $\Rightarrow$ Following Friedberg (1957) and Mučnik (1956) or Kučera (1986)

Thanks for your attention!

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Backup Interrogations

$$\frac{}{\sigma \,;\, R \vdash [\,] \,;\, [\,]} \qquad \frac{\sigma \,;\, R \vdash qs \,;\, as \qquad \sigma\, as \triangleright \text{ask } q \qquad R q a}{\sigma \,;\, R \vdash qs +\!\!+ [q] \,;\, as +\!\!+ [a]}$$

Definition (Forster, Kirst and Mück (2023))

An oracle computation is a functional $F \colon (Q \to A \to \mathbb{P}) \to I \to O \to \mathbb{P}$ captured by a computation tree $\tau \colon I \to A^* \rightharpoonup Q + O$ and its induced interrogation relation $\tau\, i \,;\, R \vdash qs \,;\, as$ as follows:

$$F\, R\, i\, o \;\leftrightarrow\; \exists qs\, as.\; \tau\, i \,;\, R \vdash qs \,;\, as \wedge \tau\, x\, as \triangleright \text{out } o$$

Backup Kleene-Post

Characterise σ_n and τ_n inductively by $\triangleright : \mathbb{N} \to \mathbb{B}^* \to \mathbb{B}^* \to \mathbb{P}$ with $0 \triangleright (\epsilon, \epsilon)$ and:

$$\frac{2n \triangleright (\sigma, \tau) \quad \sigma' \text{ least extension of } \sigma \text{ with } \chi_n\, \sigma'\, |\tau|\, b}{2n+1 \triangleright (\sigma', \tau +\!\!+ [\neg b])}$$

$$\frac{2n \triangleright (\sigma, \tau) \quad \neg(\exists \sigma' b.\, \sigma' \geq \sigma \wedge \chi_n\, \sigma'\, |\tau|\, b)}{2n+1 \triangleright (\sigma, \tau +\!\!+ [\text{false}])}$$

$$\frac{2n+1 \triangleright (\sigma, \tau) \quad \tau' \text{ least extension of } \tau \text{ with } \chi_n\, \tau'\, |\sigma|\, b}{2n+2 \triangleright (\sigma +\!\!+ [\neg b], \tau')}$$

$$\frac{2n+1 \triangleright (\sigma, \tau) \quad \neg(\exists \tau' b.\, \tau' \geq \tau \wedge \chi_n\, \tau'\, |\sigma|\, b)}{2n+2 \triangleright (\sigma +\!\!+ [\text{false}], \tau)}$$

Backup Post

Theorem

The following hold assuming Σ_n-LEM*:*

1. $\Sigma_{n+1} p$ *if and only if* $\mathcal{S}_q(p)$ *for some* q *in* Π_n*,*
2. $\Sigma_{n+1} p$ *if and only if* $\mathcal{S}_q(p)$ *for some* q *in* Σ_n*,*
3. $\Sigma_n \emptyset^{(n)}$*,*
4. $\Sigma_n p$ *implies* $p \preceq_m \emptyset^{(n)}$*, so in particular* $p \preceq_T \emptyset^{(n)}$*,*
5. $\Sigma_{n+1} p$ *if and only if* $\mathcal{S}_{\emptyset^{(n)}}(p)$*.*