Categoricity Results for Second-Order ZF in Dependent Type Theory

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Contribution

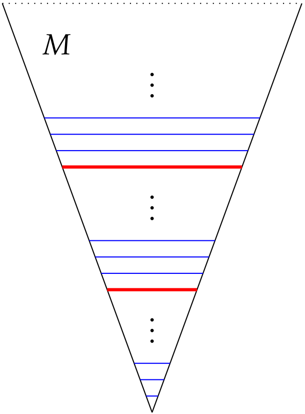
Formalisation of second-order set theory 2ZF in Coq + XM:

- Natural axiomatisation following [Barras, 2010]
- Cumulative hierarchy characterised by inductive predicate
- Zermelo's embedding theorem [Zermelo, 1930]
- Quasi-categoricity: models of 2ZF only differ in height
- Models of 2ZF have uncountable cardinality
- Grothendieck universes are inner models
- Concise development in 1500 loc (500 spec, 1000 proof)

`www.ps.uni-saarland.de/extras/itp17-sets/`

What is a model of ZF set theory?

WHAT IS A MODEL OF ZF SET THEORY?



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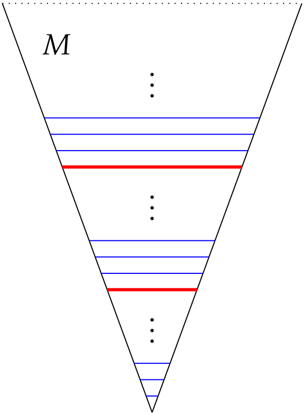

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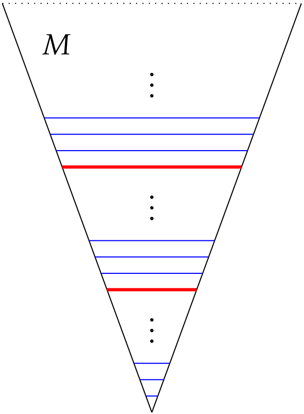

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- More sets formed by set operations (union, power, replacement, etc.)

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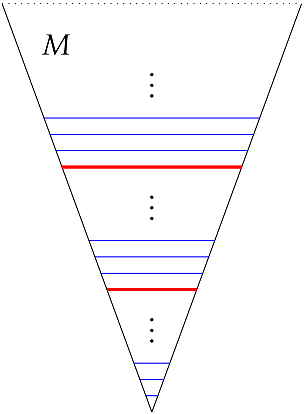

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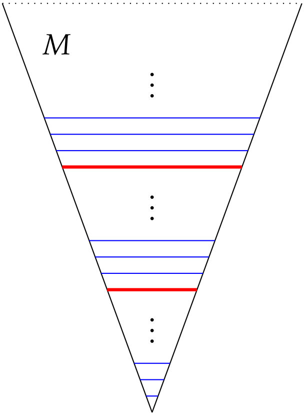

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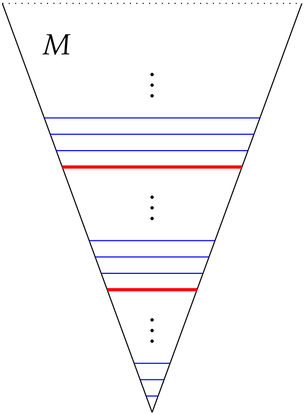

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- Only well-founded sets exist

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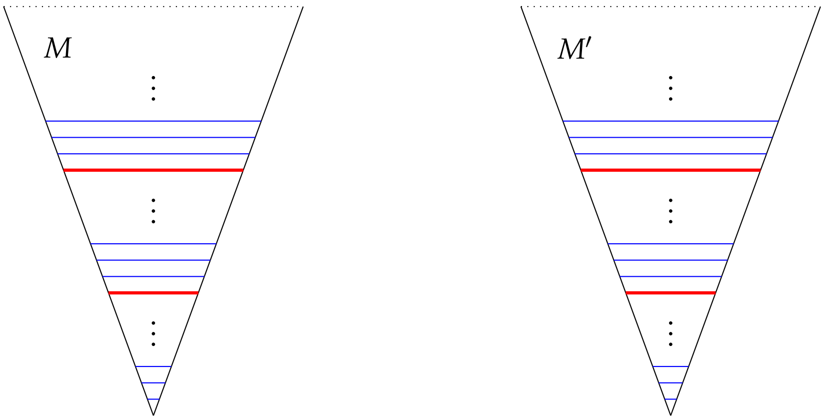

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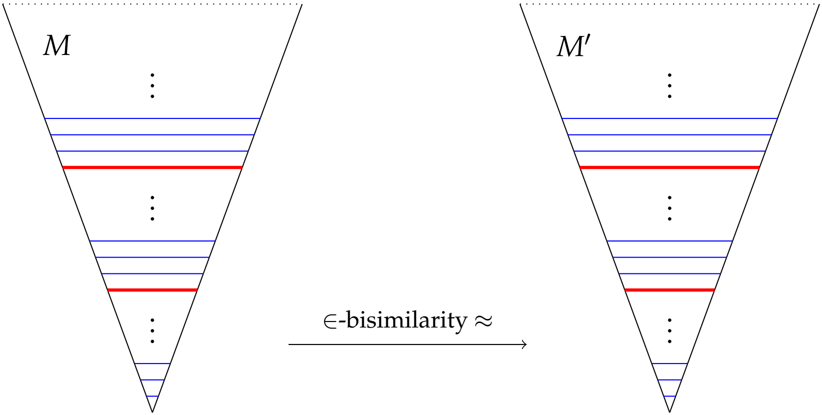

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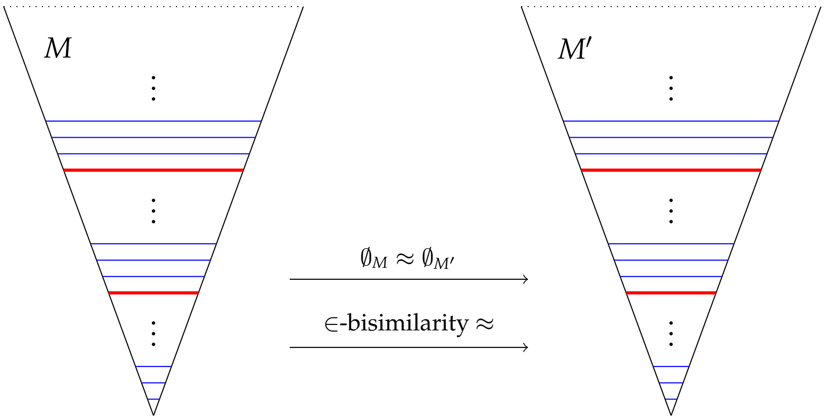

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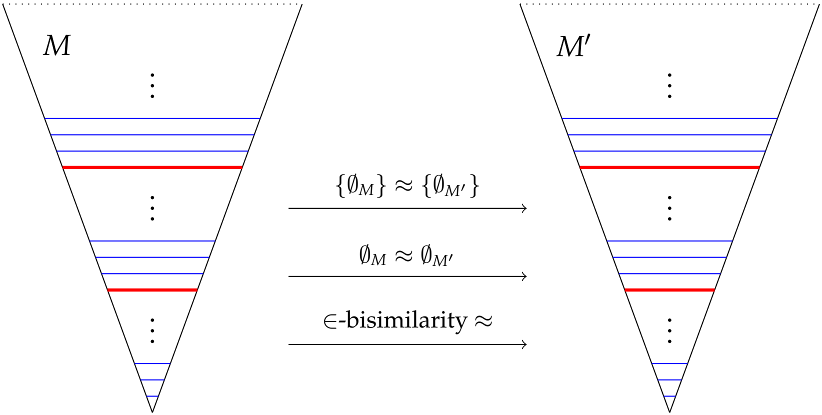

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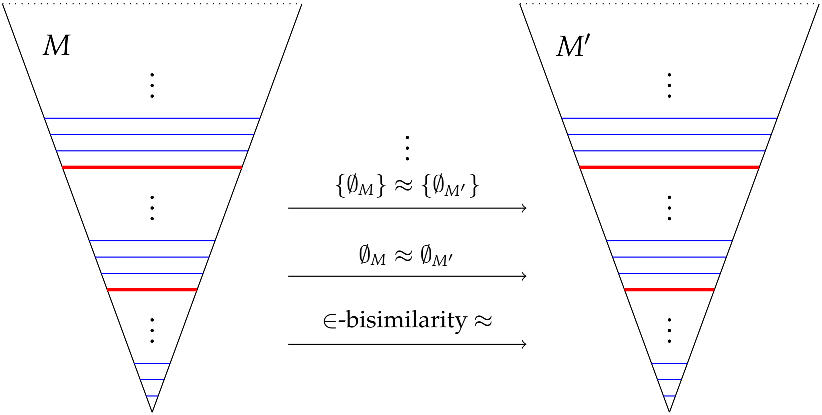

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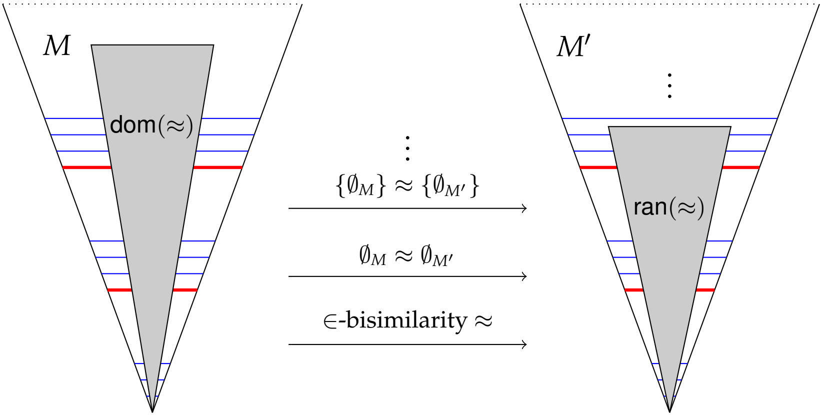

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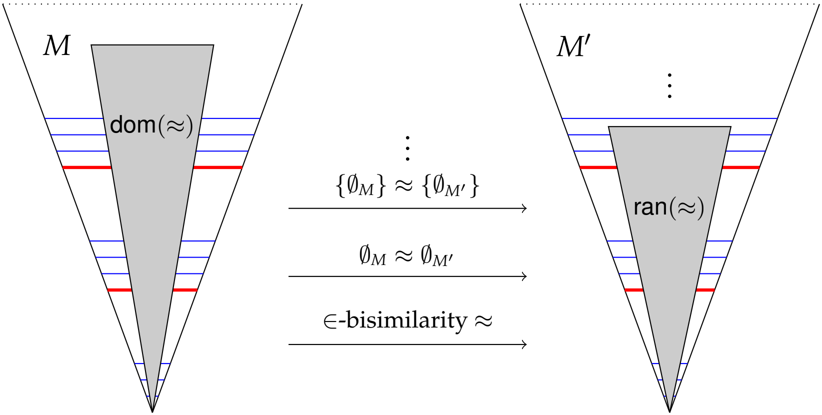

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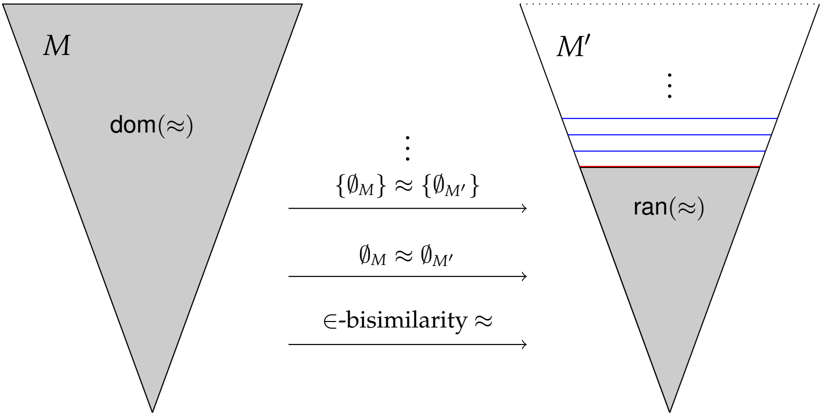

- [Skolem, 1922]: arbitrarily incompatible models in FOL
- [Zermelo, 1930]: models embed as universes in HOL

Set Structures

Definition

A **set structure** is a type M together with constants

$$\_ \in \_ : M \to M \to \mathsf{Prop} \qquad \bigcup : M \to M$$

$$\emptyset : M \qquad \mathcal{P} : M \to M$$

$$\_@\_ : (M \to M \to \mathsf{Prop}) \to M \to M$$

for membership, empty set, union, power, and replacement.

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for membership, empty set, union, power, and replacement.

Definition
We define the class of **well-founded sets** inductively by:

$$\frac{\forall y.\, y \in x \to WF\, y}{WF\, x}$$

The corresponding induction principle is called $\in$**-induction**.

Axioms System of ZF

Definition
A set structure M is a **model of ZF** if

$$\begin{aligned}
\textsf{Ext} &: \forall x, y.\, x \subseteq y \to y \subseteq x \to x = y \\
\textsf{Eset} &: \forall x.\, x \notin \emptyset \\
\textsf{Union} &: \forall x, z.\, z \in \bigcup x \leftrightarrow \exists y.\, z \in y \land y \in x \\
\textsf{Power} &: \forall x, y.\, y \in \mathcal{P}x \leftrightarrow y \subseteq x \\
\textsf{Rep} &: \forall R, x, z.\, R \in \mathcal{F}(\mathcal{M}) \to z \in R@x \leftrightarrow \exists y \in x.\, Ryz \\
\textsf{Found} &: \forall x.\, x \in WF
\end{aligned}$$

where $R \in \mathcal{F}(\mathcal{M})$ means that $R : M \to M \to \textsf{Prop}$ is functional:

$$\forall x, y, y'.\, Rxy \to Rxy' \to y = y'$$

Grothendieck Universes

Definition

A set U is called a **(Grothendieck) universe** if for all $x \in U$:

(1) $x \subseteq U$	transitivity
(2) $\emptyset \in U$	inhabitance
(3) $\bigcup x \in U$	closure under union
(4) $\mathcal{P}x \in U$	closure under power
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Fact
If M is a model and $U : M$ is a universe, then $(\Sigma x.\, x \in U)$ is a model.

Cumulative Hierarchy and Universes

Definition

We define the inductive class $\mathcal{S}$ of **stages** by the following rules:

$$\frac{\mathcal{S}\,x}{\mathcal{S}\,(\mathcal{P}x)} \qquad \frac{\forall y.\, y \in x \to \mathcal{S}\,y}{\mathcal{S}\,(\bigcup x)}$$

If a stage x satisfies $x \subseteq \bigcup x$, then we call x a **limit**.

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Sketch.

(1) Linearity by double-induction [Smullyan and Fitting, 2010], least elements and exhaustiveness by $\in$-induction.
(2) Universe U is a stage since $U = \bigcup \{ x \in U \mid \mathcal{S}\,x \}$. □

The Embedding Theorem [Zermelo, 1930]

Definition

For M, M' we define $\in$-**bisimilarity** $\approx : M \to M' \to \mathsf{Prop}$ by

$$\frac{\forall y \in x.\, \exists y' \in x'.\, y \approx y' \quad \forall y' \in x'.\, \exists y \in x.\, y \approx y'}{x \approx x'}$$

If $\approx$ is both total and surjective, we call M and M' **isomorphic**.

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Fact

The following statements hold for $x \approx x'$:

(1) $\approx$ is functional and injective

(2) $\emptyset \approx \emptyset$

(3) $\bigcup x \approx \bigcup x'$

(4) $\mathcal{P}x \approx \mathcal{P}x'$

(5) $R@x \approx \overline{R}@x'$ for $R \in \mathcal{F}(\mathcal{M})$ with $R@x \subseteq \mathsf{dom}(\approx)$

(6) $\mathsf{dom}(\approx)$ is a universe (provided it is a set)

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(1) Either M and M' are isomorphic, or

(2) $\approx$ is total and $\mathsf{ran}(\approx)$ is a universe of M', or

(3) $\approx$ is surjective and $\mathsf{dom}(\approx)$ is a universe of M.

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Sketch.

First prove $\approx$ total or surjective on stages. Then use that the stages exhaust all sets. If $\mathsf{dom}(\approx)$ or $\mathsf{ran}(\approx)$ are sets, they are universes since they reflect the original model structure. □

Categoricity Results

Fact

ZF *is categorical in every cardinality, i.e. if there is a bijection* $F : M \to M'$ *between two models, then* M *and* M' *are isomorphic.*

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Definition

$\mathbf{ZF_n}$ is **ZF** plus the existence of exactly $n : \mathbb{N}$ universes.

Fact

$\mathbf{ZF_n}$ *is categorical for every* $n : \mathbb{N}$*, i.e. if there are two models* M, M' *that satisfy* $\mathbf{ZF_n}$*, then* M *and* M' *are isomorphic.*

Future Work

- Model Constructions in Type Theory following [Aczel, 1978], [Werner, 1997] and [Barras, 2010]:
  Prove the axiomatisations $\mathbf{ZF_n}$ consistent
- Formalisation of first-order set theory:
  Independence of choice and continuum hypothesis
  by embedding of first-order syntax.
- Type-theoretic versions of cardinality results:
  Hartogs: for any type there is a larger well-ordered type
  Sierpinski: continuum hypothesis implies axiom of choice

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Linearity of Stages

Lemma (Double-Induction)

For a binary relation R on stages it holds that Rxy for all $x, y \in \mathcal{S}$ *if*
(1) $R(\mathcal{P}x)y$ *whenever Rxy and Ryx and*
(2) $R(\bigcup x)y$ *whenever Rzy for all* $z \in x$*.*

Theorem

If $x, y \in \mathcal{S}$*, then either* $x \subseteq y$ *or* $\mathcal{P}y \subseteq x$*.*

Sketch.

Apply double-induction for $Rxy \coloneqq x \subseteq y \vee \mathcal{P}y \subseteq x$. □

Development Details

File	Spec	Proof
Model.v	60	0
ST.v	139	212
Uncountable.v	21	14
Instances.v	37	66
Stage.v	95	251
Embeddding.v	163	297
Categoricity.v	9	11
Minimality.v	9	46
ZFn.v	12	28
Total	545	925