Computational Back-and-Forth Arguments in Constructive Type Theory

A Proof Pearl

Dominik Kirst
Saarland University, Saarland Informatics Campus

ITP'22, Haifa, Israel, August 9th

What is the Back-and-Forth Method?[1]

[1] See Silver's "Who invented Cantor's back-and-forth argument?" (1994) for a historic overview.

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Back-and-forth method

From Wikipedia, the free encyclopedia

In mathematical logic, especially set theory and model theory, the **back-and-forth method** is a method for showing isomorphism between countably infinite structures satisfying specified conditions. In particular it can be used to prove that

- any two countably infinite densely ordered sets (i.e., linearly ordered in such a way that between any two members there is another) without endpoints are isomorphic. An isomorphism between linear orders is simply a strictly increasing bijection. This result implies, for example, that there exists a strictly increasing bijection between the set of all rational numbers and the set of all real algebraic numbers.
- any two countably infinite atomless Boolean algebras are isomorphic to each other.
- any two equivalent countable atomic models of a theory are isomorphic.
- the Erdős–Rényi model of random graphs, when applied to countably infinite graphs, always produces a unique graph, the Rado graph.
- any two many-complete recursively enumerable sets are recursively isomorphic.

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General technique to establish isomorphisms for countable structures:

- Any two countable unbounded dense linear orders are isomorphic (Cantor (1895))
- Any two one-one interreducible sets are recursively isomorphic (Myhill (1957))
- Many more examples in model theory, Boolean algebras, and graph theory

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- Instantiation to Cantor's and Myhill's isomorphism theorems
- Observations about the computational interpretation of the results
- Mechanisation of all results in Coq:

`www.ps.uni-saarland.de/extras/back-and-forth`

The Informal Argument: Cantor's Isomorphism Theorem

Assume two countable unbounded dense linear orders $(X, <)$ and $(Y, <)$, think $(\mathbb{Q}, <)$.

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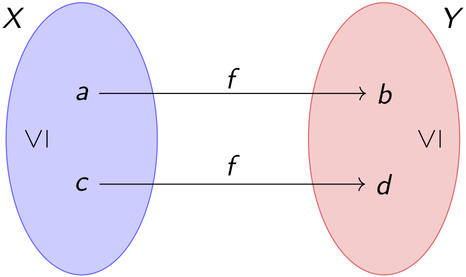
Given a partial order-isomorphism $f : X \to Y$, how do we extend it to a new element x?



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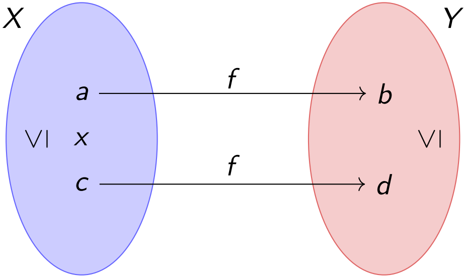

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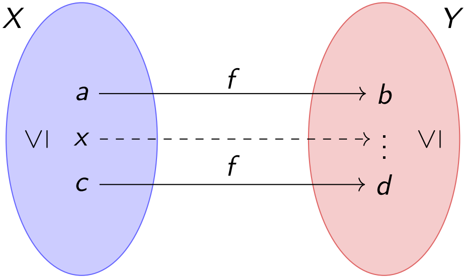

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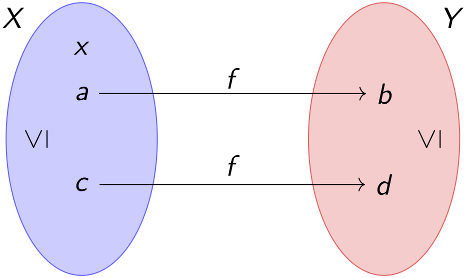

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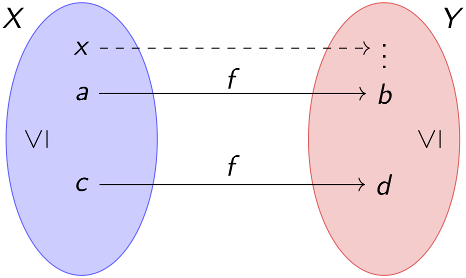

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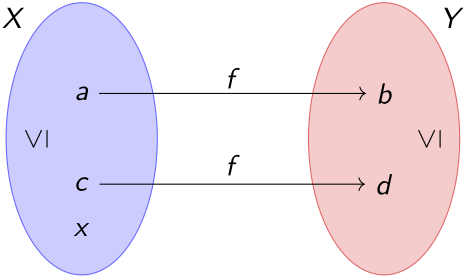

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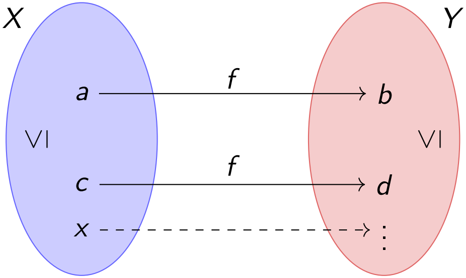

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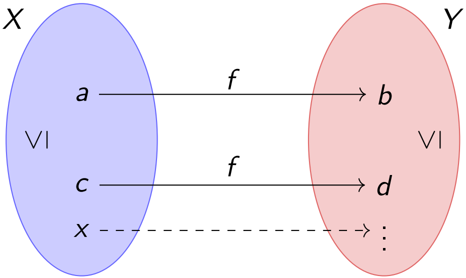

Obtain a total order-isomorphism F:

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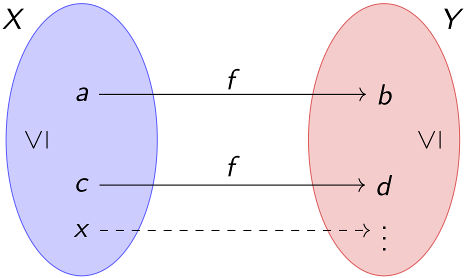

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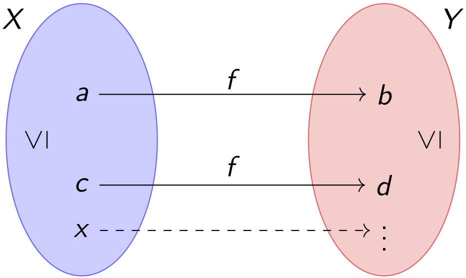

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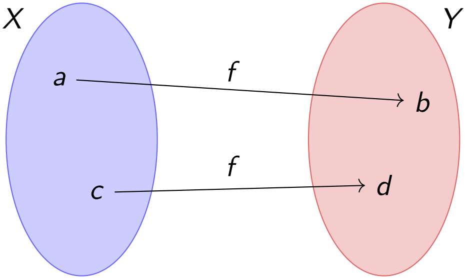

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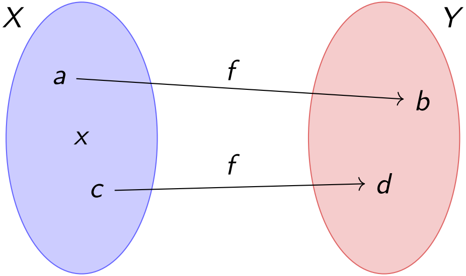

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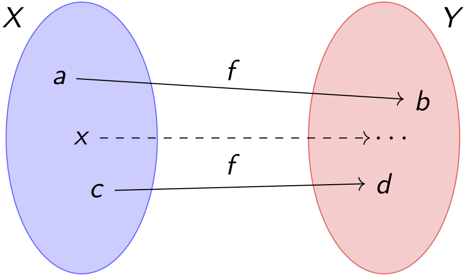

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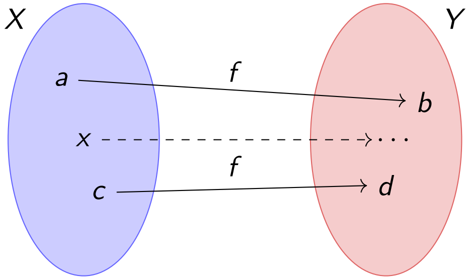

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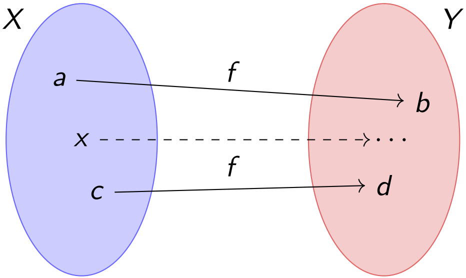

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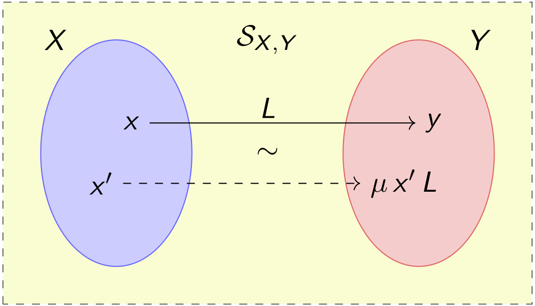

Assumed data:

- Countable types $X, Y : \mathbb{T}$
- Abstract structure $\mathcal{A} : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$
- Structure $\mathcal{S}_{X,Y} : \mathcal{A}(X, Y)$ on X and Y
- $\sim : (X \times X) \to (Y \times Y) \to \mathbb{P}$
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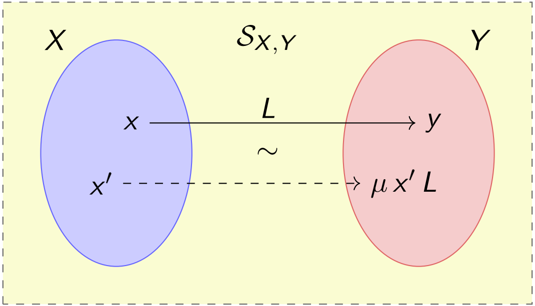

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- If $(x, x') \sim (y, y')$, then $x = x'$ iff $y = y'$.
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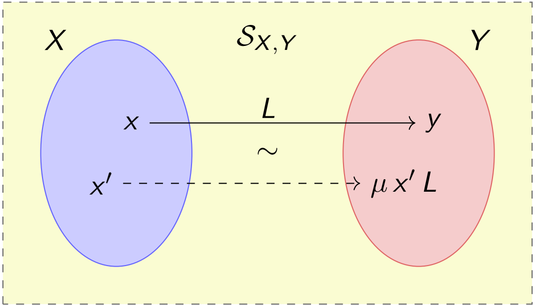

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L_{\_} &: \forall XY.\, \mathcal{A}(X, Y) \to \mathbb{N} \to \mathcal{L}(X \times Y) \\
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Goal: F preserves the abstract structure, i.e. $\forall xx'.\, (x,x') \sim (F\,x, F\,x')$.

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Theorem (Isomorphism)

F satisfies $(x, x') \sim (F\,x, F\,x')$ for all $x, x' : X$ and is inverted by F^{-1}.

The Abstract Argument in Coq

```
(* Assumptions regarding the invertable abstract structure 𝒜 *)
struc : Type → Type → Type
srev : ∀ X Y, struc X Y → struc Y X
srev_invol : ∀ X Y (S : struc X Y), srev (srev S) = S

(* Assumptions regarding the abstract isomorphism property ∼ *)
iso : ∀ X Y, struc X Y → X → X → Y → Y → Prop
iso_eq : ∀ X Y (S : struc X Y) x x' y y', iso S x x' y y' → x = x' ↔ y = y'
iso_rev : ∀ X Y (S : struc X Y) x x' y y', iso S x x' y y' → iso (srev S) y y' x x'

(* Assumptions regarding the one-step extension function μ *)
find : ∀ X Y, struc X Y → X → list (X * Y) → Y
find_iso : ∀ X Y (S : struc X Y) L x, x ∉ dom L → tiso S L → tiso S ((x, find S x L) :: L)

(* After 100 lines of abstract proofs: *)
Theorem back_and_forth :
  { F & { G | inverse F G ∧ ∀ x x', iso SXY x x' (F x) (F x') } }.
```

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Computational interpretation: enumerable and discrete structures with a structure-preserving extension algorithm are computably isomorphic.

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There is a function μ such that if L is a partial order-isomorphism, so is $(x, \mu\, x\, L) :: L$.

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Cantor's Isomorphism Theorem[3]

$$\text{Structure } \mathcal{A}(X, Y) \;:=\; X \text{ and } Y \text{ are unboundend dense linear orders}$$
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All countable unbounded dense linear orders are isomorphic.

Theorem (Computational Cantor)

All decidable linear orders over enumerable domain with computable witnesses for density and unboundedness are computably isomorphic.

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Cantor's Isomorphism Theorem in Coq

```
(* After 150 lines to define partner and prove step_morph: *)
Theorem Cantor X Y (OX : dulo X) (OY : dulo Y) (RX : retract X nat) (RY : retract Y nat) :
  { F : X → Y & { G | inverse F G ∧ ∀ x x', x < x' ↔ (F x) < (F x') } }.
Proof.
  unshelve edestruct back_and_forth as [F[G[H1 H2]]].
  - intros A B. exact (dulo A * dulo B).                                (* structure *)
  - intros A B [OA OB]. exact (OB, OA).                                 (* srev *)
  - cbn. intros A B [OA OB]. reflexivity.                                (* srev_invol *)
  - intros A B [OA OB] a a' b b'. exact ((a = a' ↔ b = b') ∧ (a < a' ↔ b < b')).   (* iso *)
  - cbn. tauto.                                                          (* iso_eq *)
  - cbn. tauto.                                                          (* iso_rev *)
  - cbn. intros A B [OA OB] f a. exact (partner OA OB f a).              (* find *)
  - cbn. intros A B [OA OB] f x. unfold step, tiso. now apply step_morph.   (* find_iso *)

  - exact X.
  - exact Y.
  - exact (OX, OY).
  - exact RX.
  - exact RY.

  - cbn in *. exists F, G. split; try apply H1. apply H2.
Qed.
```

Myhill's Isomorphism Theorem[4]

"One-one interreducible sets of numbers are recursively isomorphic."

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Myhill's Isomorphism Theorem[4]

"One-one interreducible sets of numbers are recursively isomorphic."

A function $f : X \to Y$ is a many-one reduction from $p : X \to \mathbb{P}$ to $q : Y \to \mathbb{P}$ if

$$\forall x : X.\, p\, x \leftrightarrow q\,(f\, x).$$

If f is injective (bijective), it is a one-one reduction (recursive isomorphism).

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$$\begin{aligned} \text{Structure } \mathcal{A}(X, Y) &:= \text{ a pair } (p, q) \text{ of one-one interreducible predicates} \\ (x, x') \sim (y, y') &:= (x = x' \leftrightarrow y = y') \wedge (p\,x \leftrightarrow q\,y) \end{aligned}$$

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Theorem (Myhill)

One-one interreducible unary predicates on (retracts of) $\mathbb{N}$ are recursively isomorphic.

[4] Discovered by Myhill (1957), previously mechanised by Forster, Jahn, and Smolka (2022).

Myhill's Isomorphism Theorem in Coq

```
(* After 100 lines to define mstep and prove step_corr: *)
Theorem Myhill X Y (SXY : bireduction X Y) (RX : retract X nat) (RY : retract Y nat) :
  { F : X → Y & { G | inverse F G ∧ reduction p q F } }.
Proof.
  unshelve edestruct back_and_forth as [F[G[H1 H2]]].
  - intros A B. exact (bireduction A B).                                       (* structure *)
  - intros A B S. cbn in *. apply (@Build_bireduction B A eY eX q p g f); apply S.   (* srev *)
  - cbn. intros A B []. reflexivity.                                           (* srev_invol *)
  - intros A B S a a' b b'. cbn in S. exact ((a = a' ↔ b = b') ∧ (p a ↔ q b)).  (* iso *)
  - cbn. tauto.                                                                (* iso_eq *)
  - cbn. tauto.                                                                (* iso_rev *)
  - cbn. intros A B S C a. exact (mstep f eX eY C a).                          (* find *)
  - cbn. intros A B S C a. unfold step, tiso. apply step_corr; apply S.        (* find_iso *)

  - exact X.
  - exact Y.
  - exact SXY.
  - exact RX.
  - exact RY.

  - cbn in *. exists F, G. split; try apply H1. intros x. now apply H2.
Qed.
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Bonus: Computational Cantor-Bernstein Theorem[5]

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If we forget about the computational interpretation of the previous theorem, we can observe:

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If we forget about the computational interpretation of the previous theorem, we can observe:

Corollary (Countable Cantor-Bernstein)

Countable sets X and Y with injections $X \to Y$ and $Y \to X$ admit a bijection $X \to Y$.

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Pick as p and q the full predicates on X and Y, then use the previous theorem. □

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Though Pradic and Brown (2019) show that the general Cantor-Bernstein theorem is equivalent to excluded middle, the restriction to countable sets holds constructively!

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Thanks for listening!

Bibliography

Bauer, A. (2006). First steps in synthetic computability theory. *Electronic Notes in Theoretical Computer Science*, 155:5–31.

Cantor, G. (1895). Beiträge zur Begründung der transfiniten Mengenlehre. *Mathematische Annalen*, 46(4):481–512.

Forster, Y., Jahn, F., and Smolka, G. (2022). A Constructive and Synthetic Theory of Reducibility: Myhill's Isomorphism Theorem and Post's Problem for Many-one and Truth-table Reducibility in Coq (Full Version). working paper or preprint.

Forster, Y., Kirst, D., and Smolka, G. (2019). On synthetic undecidability in Coq, with an application to the entscheidungsproblem. In *Proceedings of the 8th ACM SIGPLAN International Conference on Certified Programs and Proofs*, pages 38–51.

Giese, M. and Schönegge, A. (1995). *Any Two Countable, Densely Ordered Sets Without Endpoints are Isomorphic: A Formal Proof with KIV*. Univ., Fak. für Informatik.

Marzion, E. (2020). Visualizing Cantor's theorem on dense linear orders using Coq. Blog post.

Myhill, J. (1957). Creative sets. *Journal of Symbolic Logic*, 22(1).

Pradic, P. and Brown, C. E. (2019). Cantor-Bernstein implies excluded middle. *arXiv preprint arXiv:1904.09193*.

Richman, F. (1983). Church's thesis without tears. *The Journal of symbolic logic*, 48(3):797–803.

Silver, C. L. (1994). Who invented Cantor's back-and-forth argument? *Modern Logic*, 4(1):74–78.

Synthetic Computability Theory

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Myhill Construction

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