Formalising Metamathematics in Constructive Type Theory

Synthetic Undecidability and Incompleteness

Dominik Kirst

Proof and Computation
September 14th, 2021

SAARLAND UNIVERSITY
COMPUTER SCIENCE
SIC Saarland Informatics Campus

What will this talk be about?

Metamathematics (of first-order logic):

- Mostly negative results: undecidability and incompleteness[*]
- Sketch positive results: completeness and (relative) consistency[†]

Constructive type theory:

- Basic concepts of the calculus of inductive constructions (CIC)[‡]
- Implementation in the Coq proof assistant[§]
- Synthetic computability[¶]

---

[*] Tarski (1953); Gödel (1931)
[†] Gödel (1930); Werner (1997)
[‡] Coquand and Huet (1986); Paulin-Mohring (1993)
[§] The Coq Development Team (2021)
[¶] Richman (1983); Bauer (2006)

Outline

- Framework: Synthetic Undecidability
- Example 1: The Entscheidungsproblem
- Example 2: Trakhtenbrot's Theorem
- Example 3: First-Order Axiom Systems
- Conclusion

Framework: Synthetic Undecidability*

*Yannick Forster, K., and Gert Smolka at CPP'19.

How to mechanise decidability?

Conventional approach:

- Pick a concrete model of computation
  (Turing machines, μ-recursive functions, untyped λ-calculus, etc.)
- Invent a decision procedure for the given problem
- Explicitly code the algorithm in the chosen model!

Synthetic approach (Richman (1983); Bauer (2006)):

- Work in a constructive foundation, e.g. constructive type theory
- Define a decision procedure e.g. as a Boolean function
- Definable functions are computable, so that's it!

(Similar for other notions like enumerability and reducibility)

How to mechanise undecidability?

Problem of the synthetic approach:

- Constructive type theories like CIC are consistent with classical assumptions, rendering every problem decidable
- Proving a given problem undecidable is not outright possible

Possible solutions:

- Resort to a concrete model of computation
- Verify a synthetic reduction from an undecidable problem
  - Computability axioms could be used to obtain expected results

(Again similar for other negative notions of computability theory)

Coq's Type Theory

Main features of Coq's underlying CIC:

- Standard type formers: $X \to Y$, $X \times Y$, $X + Y$, $\forall x.\, F\,x$, $\Sigma x.\, F\,x$
- Inductive types: $\mathbb{B}$, $\mathbb{N}$, lists $\mathcal{L}(X)$, options $\mathcal{O}(X)$, vectors X^n, ...
- Propositional universe $\mathbb{P}$ with logical connectives: $\to$, $\land$, $\lor$, $\forall$, $\exists$
- $\mathbb{P}$ is impredicative and separate from computational types

All definable functions $\mathbb{N} \to \mathbb{N}$ are computable!

Decidability and Enumerability

A problem interpreted as a predicate $p : X \to \mathbb{P}$ on a type X is decidable if there is a function $f : X \to \mathbb{B}$ with

$$\forall x.\, p\, x \leftrightarrow f\, x = \mathsf{tt},$$

enumerable if there is a function $f : \mathbb{N} \to \mathcal{O}(X)$ with

$$\forall x.\, p\, x \leftrightarrow \exists n.\, f\, n = \ulcorner x \urcorner.$$

Fact

Let $p : X \to \mathbb{P}$ be a predicate, then p is

- *decidable iff $\forall x.\, p\, x + \neg p\, x$ is inhabited and*
- *enumerable iff there is $L : \mathbb{N} \to \mathcal{L}(X)$ s.t. $\forall x.\, p\, x \leftrightarrow \exists n.\, x \in L\, n$.*

Data Types

Computability theory is usually developed on computational domains.

A type X is called

- enumerable if $\lambda x.\, \top$ is enumerable,
- discrete if $\lambda xy.\, x = y$ is decidable, and
- data type if it is both enumerable and discrete.

Fact

Decidable predicates p on data types X are enumerable and co-enumerable.

Proof.

Let $f_X : \mathbb{N} \to \mathcal{O}(X)$ enumerate X and $f_p : X \to \mathbb{B}$ decide p. Then

$$f\ n := \text{match } f_X\ n \text{ with } \ulcorner x \urcorner \Rightarrow \text{if } f_p\, x \text{ then } \ulcorner x \urcorner \text{ else } \emptyset \mid \emptyset \Rightarrow \emptyset$$

defines an enumerator for p. □

Post's Theorem

Theorem

Let p on a data type X be enumerable and co-enumerable. If p is also logically decidable, i.e. $\forall x.\, p\,x \vee \neg p\,x$, then it is decidable.

Proof.

- Let f enumerate p and g enumerate its complement $\overline{p}$.
- $\forall x.\, \exists n.\, f\,n = \ulcorner x \urcorner \vee g\,n = \ulcorner x \urcorner$ by logical decidability.
- For given x, corresponding n can be computed by linear search.
- Disjunction $f\,n = \ulcorner x \urcorner \vee g\,n = \ulcorner x \urcorner$ lacks computational information.
- Use discreteness to computably compare $\ulcorner x \urcorner$ with $f\,n$ and $g\,n$.
- Obtain decision whether $p\,x$ or $\neg p\,x$. $\square$

Many-One Reductions

Given predicates $p : X \to \mathbb{P}$ and $q : Y \to \mathbb{P}$ we call a function $f : X \to Y$ a (many-one) reduction from p to q if

$$\forall x.\, p\, x \leftrightarrow q\,(f\, x).$$

We write $p \preccurlyeq q$ if a reduction from p to q exists.

Theorem (Reduction)

Let p and q be predicates on data types with $p \preccurlyeq q$.

- *If q is decidable/enumerable/co-enumerable, then so is p.*
- *If p is not co-enumerable, then q is not co-enumerable.*

Proof.

If f witnesses $p \preccurlyeq q$ and g decides q, then $g \circ f$ decides p. $\square$

The Post Correspondence Problem

Intuition: given a stack S of cards s/t, find a derivable match.

This (undecidable) problem can be expressed by an inductive predicate:

$$\frac{s/t \in S}{S \triangleright s/t} \qquad \frac{S \triangleright u/v \quad s/t \in S}{S \triangleright su/tv} \qquad \frac{S \triangleright s/s}{\mathrm{PCP}\ S}$$

Fact

The type $\mathbb{S}$ of stacks is a data type and PCP *is enumerable.*

Proof.

The former follows from closure properties and for the latter

$$\begin{aligned} L\,0 &:= [] \\ L\,(\mathrm{S}\,n) &:= L\,n \mathbin{+\!\!+} [(S,(s,t)) \mid S \in L_{\mathbb{S}}\,n, (s,t) \in S] \\ &\qquad \mathbin{+\!\!+} [(S,(su,tv)) \mid (S,(u,v)) \in L\,n, (s,t) \in S] \end{aligned}$$

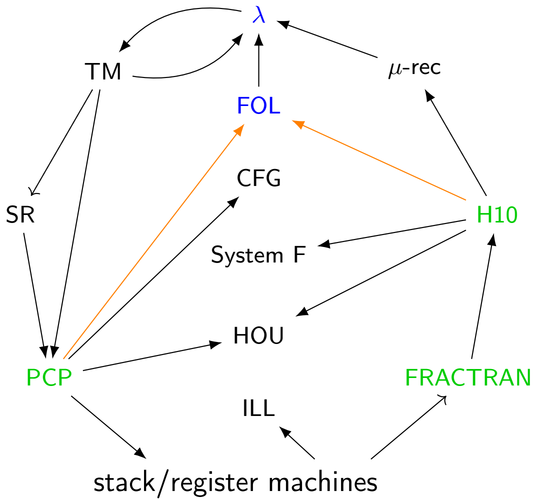
defines a list enumerator for $\lambda Sst.\, S \triangleright s/t$. $\square$

Coq Library of Undecidability Proofs*

- Merge of a few initial Coq developments:
  - Computablity theory using a cbv. lambda calculus
  - Synthetic computability
  - Initial undecidability proofs
- Extended with further undecidability reductions over past 3 years
- Unified framework to ease external contribution
- 11+ contributors and more than 100k lines of code
- 14+ related publications (ITP, CPP, IJCAR, FSCD, etc.)
- Currently roughly 13 (groups of) undecidable problems

*https://github.com/uds-psl/coq-library-undecidability

Library Overview (Forster et al. (2020b))



- Classification in seed problems and target problems
- This talk: mostly the PCP $\rightarrow$ FOL edge, a bit of H10 $\rightarrow$ FOL

Example 1: The Entscheidungsproblem*

*Yannick Forster, K., and Gert Smolka at CPP'19.

General Idea

Given a FOL formula φ, is φ valid in all models?

Conventional outline following Turing:

- Encode Turing machine M as formula φ_M over custom signature
- Verify that M halts if and only if φ_M holds in all models
- (Argue why the used signature could have been minimised)

Our outline:

- Follow the simpler proof given in Manna (2003) using PCP
- Also don't bother with signature minimisation yet...

Syntax and Tarski Semantics

Terms and formulas are defined for a fixed signature:

$$\tau : \mathsf{Term} := x \mid a \mid e \mid f_{\mathsf{tt}}\,\tau \mid f_{\mathsf{ff}}\,\tau \qquad x, a : \mathbb{N}$$
$$\varphi, \psi : \mathsf{Form} := \dot{\bot} \mid Q \mid P\,\tau_1\,\tau_2 \mid \varphi \dot{\rightarrow} \psi \mid \dot{\forall} x.\,\varphi$$

Formulas are interpreted in models $\mathcal{I} = (D, \eta, e^{\mathcal{I}}, f_{\mathsf{tt}}^{\mathcal{I}}, f_{\mathsf{ff}}^{\mathcal{I}}, Q^{\mathcal{I}}, P^{\mathcal{I}})$ given a variable environment $\rho : \mathbb{N} \to D$:

$$\begin{aligned}
\mathcal{I} \vDash_\rho \dot{\bot} &:= \bot \\
\mathcal{I} \vDash_\rho Q &:= Q^{\mathcal{I}} \\
\mathcal{I} \vDash_\rho P\,\tau_1\,\tau_2 &:= P^{\mathcal{I}}\,(\hat{\rho}\,\tau_1)\,(\hat{\rho}\,\tau_2) \\
\mathcal{I} \vDash_\rho \varphi \dot{\rightarrow} \psi &:= \mathcal{I} \vDash_\rho \varphi \rightarrow \mathcal{I} \vDash_\rho \psi \\
\mathcal{I} \vDash_\rho \dot{\forall} x.\,\varphi &:= \forall d : D.\, \mathcal{I} \vDash_{\rho[x:=d]} \varphi
\end{aligned}$$

A formula φ is valid if $\mathcal{I} \vDash_\rho \varphi$ for all $\mathcal{I}$ and ρ.

A Standard Model

Strings can be encoded as terms, e.g. $\overline{\mathsf{tt}\,\mathsf{ff}\,\mathsf{ff}\,\mathsf{tt}} = f_{\mathsf{tt}}\,(f_{\mathsf{ff}}\,(f_{\mathsf{ff}}\,(f_{\mathsf{tt}}\,(e))))$.

The standard model $\mathcal{B}$ over the type $\mathcal{L}(\mathbb{B})$ of Boolean strings captures exactly the cards derivable from a fixed stack S:

$$e^{\mathcal{B}} := [] \qquad\qquad Q^{\mathcal{B}} := \mathsf{PCP}\,S$$

$$f_b^{\mathcal{B}}\,s := b :: s \qquad\qquad P^{\mathcal{B}}\,s\,t := S \triangleright s/t.$$

Lemma

Let $\rho : \mathbb{N} \to \mathcal{L}(\mathbb{B})$ be an environment for the standard model $\mathcal{B}$.
Then $\hat{\rho}\,\overline{s} = s$ and $\mathcal{B} \vDash_\rho P\,\tau_1\,\tau_2 \leftrightarrow S \triangleright \hat{\rho}\,\tau_1/\hat{\rho}\,\tau_2$.

Undecidability of Validity

We express the constructors of $S \triangleright s/t$ and PCP as formulas:

$$\varphi_1 := [P\,\overline{s}\,\overline{t} \mid s/t \in S]$$
$$\varphi_2 := [\dot\forall xy.\, P\,x\,y \dot\to P\,(\overline{s}x)\,(\overline{t}y) \mid s/t \in S]$$
$$\varphi_3 := \dot\forall x.\, P\,x\,x \dot\to Q$$
$$\varphi_S := \varphi_1 \dot\to \varphi_2 \dot\to \varphi_3 \dot\to Q$$

Theorem

PCP S *iff* φ_S *is valid, hence* PCP *reduces to validity.*

Proof.

Let φ_S be valid, so in particular $\mathcal{B} \vDash \varphi_S$. Since $\mathcal{B}$ satisfies all of φ_1, φ_2, and φ_3 it follows that $\mathcal{B} \vDash Q$ and thus PCP S.
Now suppose that $S \triangleright s/s$ for some s and that some model $\mathcal{I}$ satisfies all of φ_1, φ_2, and φ_3. Then $\mathcal{I} \vDash P\,\overline{s}\,\overline{s}$ by φ_1 and φ_2, hence $\mathcal{I} \vDash Q$ by φ_3, and thus $\mathcal{I} \vDash \varphi_S$. □

Undecidability of Satisfiability

Disclaimer: validity does not directly reduce to (co-)satisfiability!

- If φ is valid, then certainly $\dot\neg\varphi$ is unsatisfiable
- However, the converse does not hold constructively

Fortunately, we can give a direct reduction from the complement of PCP:

Theorem

$\neg\mathsf{PCP}\,S$ *iff* $\dot\neg\varphi_S$ *is satisfiable, hence co-*PCP *reduces to satisfiability.*

Proof.

If $\neg\mathsf{PCP}\,S$, then $\mathcal{B} \vDash \dot\neg\varphi_S$ since $\mathcal{B} \vDash \varphi_S$ would yield $\mathsf{PCP}\,S$ as before. Now suppose there are $\mathcal{I}$ and ρ with $\mathcal{I} \vdash_\rho \dot\neg\varphi_S$. Then assuming $\mathsf{PCP}\,S$ yields the contradiction that φ_S is valid. $\square$

Interlude: Completeness Theorems for FOL

Completeness of deduction systems for FOL relies on Markov's principle:

$$\mathsf{MP} := \forall f : \mathbb{N} \to \mathbb{B}.\, \neg\neg(\exists n.\, f\ n = \mathsf{tt}) \to \exists n.\, f\ n = \mathsf{tt}$$

MP is independent but admissible in Coq's type theory*

Theorem (cf. Yannick Forster, K., and Dominik Wehr at LFCS'20.)

- $\mathcal{T} \vDash \varphi$ *implies* $\neg\neg(\mathcal{T} \vdash_c \varphi)$ *for all* $\mathcal{T} : \mathsf{Form} \to \mathbb{P}$ *and* $\varphi : \mathsf{Form}$
- *If* $\mathcal{T}$ *is enumerable, then* MP *is equivalent to the stability of* $\mathcal{T} \vdash_c \varphi$

$\Rightarrow$ *Completeness for enumerable* $\mathcal{T}$ *is equivalent to* MP *and admissible*

Possible strategies:

a) Verify a weak reduction from PCP integrating the double negation
b) Obtain a standard reduction by proving $A \vdash_c \varphi_S$ by hand (done so far)

*Coquand/Mannaa '17, Pédrot/Tabareau '18

Undecidability of Minimal Provability

We define a minimal natural deduction system inductively:

$$\frac{\varphi \in A}{A \vdash \varphi}\ A \qquad \frac{\varphi :: A \vdash \psi}{A \vdash \varphi \dot{\to} \psi}\ II \qquad \frac{A \vdash \varphi \dot{\to} \psi \quad A \vdash \varphi}{A \vdash \psi}\ IE$$

$$\frac{A \vdash \varphi^x_a \quad a \notin \mathcal{P}(\varphi) \cup \mathcal{P}(A)}{A \vdash \dot{\forall} x.\, \varphi}\ AI \qquad \frac{A \vdash \dot{\forall} x.\, \varphi \quad \mathcal{V}(\tau) = \emptyset}{A \vdash \varphi^x_\tau}\ AE$$

A formula φ is provable if $\vdash \varphi$.

Fact (Soundness)

$A \vdash \varphi$ implies $A \vDash \varphi$, so provable formulas are valid.

Theorem

- PCP S *iff φ_S is provable.* *(proving $\vdash \varphi_S$ by hand)*
- *Provability is enumerable.* *(by giving a list enumerator)*

Undecidability of Classical Provability

We extend the deduction system by classical double negation elimination:

$$\frac{A \vdash_c \dot{\neg}\dot{\neg}\varphi}{A \vdash_c \varphi}\ DN$$

Unfortunately, this rule is not sound constructively!

As a remedy, we define a Gödel-Gentzen-Friedman translation φ^Q of formulas φ such that $A \vdash_c \varphi$ implies $A^Q \vdash \varphi^Q$.

Theorem

PCP S *iff* φ_S *is classically provable, hence* PCP *reduces to classical ND.*

Proof.

If PCP S then $\vdash \varphi_S$ by the previous theorem and hence $\vdash_c \varphi_S$. Conversely, let $\vdash_c \varphi_S$ and hence $\vdash \varphi_S^Q$. Then by soundness $\mathcal{B} \vDash \varphi_S^Q$ which still implies $\mathcal{B} \vDash Q$ and PCP S as before. □

Example 2: Trakhtenbrot's Theorem*

*K. and Dominique Larchey-Wendling at IJCAR'20.

General idea

Given a FOL formula φ, is φ finitely satisfiable?

Textbook proofs by dual reduction from the halting problem:*

- Encode Turing machine M as formula φ_M over custom signature
- Verify that the models of φ_M correspond to the runs of M
- Conclude that M halts if and only if φ_M has a finite model

Our mechanisation:

- Illustrates that one can still use PCP for a simpler reduction
- Signature minimisations are constructive for finite models

*e.g. Libkin (2010); Börger et al. (1997)

First-Order Satisfiability over Signatures

Given a signature $\Sigma = (\mathcal{F}_\Sigma; \mathcal{P}_\Sigma)$, we represent terms and formulas by:

$$t \;:\; \mathsf{Term}_\Sigma \;::=\; x \mid f\,\vec{t} \qquad\qquad (x : \mathbb{N},\ f : \mathcal{F}_\Sigma,\ \vec{t} : \mathsf{Term}_\Sigma^{|f|})$$

$$\varphi, \psi \;:\; \mathsf{Form}_\Sigma \;::=\; \dot{\bot} \mid P\,\vec{t} \mid \varphi \,\dot{\Box}\, \psi \mid \dot{\nabla}\varphi \qquad\qquad (P : \mathcal{P}_\Sigma,\ \vec{t} : \mathsf{Term}_\Sigma^{|P|})$$

A model $\mathcal{M}$ over a domain D is a pair of interpretation functions:

$$\_^{\mathcal{M}} \;:\; \forall f : \mathcal{F}_\Sigma.\, D^{|f|} \to D \qquad\qquad \_^{\mathcal{M}} \;:\; \forall P : \mathcal{P}_\Sigma.\, D^{|P|} \to \mathbb{P}$$

For assignments $\rho : \mathbb{N} \to D$ define evaluation $\hat{\rho}\, t$ and satisfaction $\mathcal{M} \vDash_\rho \varphi$:

$$\hat{\rho}\, x \;:=\; \rho x \qquad\qquad \hat{\rho}\,(f\,\vec{t}) \;:=\; f^{\mathcal{M}}\,(\hat{\rho}\,\vec{t})$$

$$\mathcal{M} \vDash_\rho \dot{\bot} \;:=\; \bot \qquad\qquad \mathcal{M} \vDash_\rho \varphi \,\dot{\Box}\, \psi \;:=\; \mathcal{M} \vDash_\rho \varphi \,\Box\, \mathcal{M} \vDash_\rho \psi$$

$$\mathcal{M} \vDash_\rho P\,\vec{t} \;:=\; P^{\mathcal{M}}\,(\hat{\rho}\,\vec{t}) \qquad\qquad \mathcal{M} \vDash_\rho \dot{\nabla}\varphi \;:=\; \nabla a : D.\, \mathcal{M} \vDash_{a \cdot \rho} \varphi$$

$\mathsf{SAT}(\Sigma)\,\varphi :=$ there are $\mathcal{M}$ and ρ such that $\mathcal{M} \vDash_\rho \varphi$

Finiteness in Constructive Type Theory

Definition

A type X is finite if there exists a list l_X with $x \in l_X$ for all $x : X$.

This seems to be a good compromise:

- Easy to establish and work with
- Does not enforce discreteness
- Enough to get expected properties:
  - Every strict order on a finite type is well-founded
  - Every finite decidable equivalence relation admits a quotient on $\mathbb{F}_n$

$\mathsf{FSAT}(\Sigma)\,\varphi$ if additionally D is finite and all $P^{\mathcal{M}}$ are decidable

$\mathsf{FSATEQ}(\Sigma; \equiv)\,\varphi$ if $x \equiv^{\mathcal{M}} y \leftrightarrow x = y$ for all $x, y : D$ (hence discrete)

Encoding the Post Correspondence Problem

We use the signature $\Sigma_{\mathsf{BPCP}} := (\{\star^0, e^0, f^1_{\mathsf{tt}}, f^1_{\mathsf{ff}}\}; \{P^2, \prec^2, \equiv^2\})$:

- Chains like $f_{\mathsf{ff}}(f_{\mathsf{tt}}(e))$ represent strings while $\star$ signals overflow
- P concerns only defined values and $\prec$ is a strict ordering:

$$
\begin{aligned}
\varphi_P &:= \dot\forall xy.\, P\,x\,y \mathrel{\dot\to} x \not\equiv \star \mathrel{\dot\land} y \not\equiv \star \\
\varphi_\prec &:= (\dot\forall x.\, x \not\prec x) \mathrel{\dot\land} (\dot\forall xyz.\, x \prec y \mathrel{\dot\to} y \prec z \mathrel{\dot\to} x \prec z)
\end{aligned}
$$

- Sanity checks on f regarding overflow, disjointness, and injectivity:

$$
\varphi_f \;:=\; \begin{pmatrix} f_{\mathsf{tt}}\,\star \equiv \star \mathrel{\dot\land} f_{\mathsf{ff}}\,\star \equiv \star \\ \dot\forall x.\, f_{\mathsf{tt}}\,x \not\equiv e \\ \dot\forall x.\, f_{\mathsf{ff}}\,x \not\equiv e \end{pmatrix} \mathrel{\dot\land} \begin{pmatrix} \dot\forall xy.\, f_{\mathsf{tt}}\,x \not\equiv \star \mathrel{\dot\to} f_{\mathsf{tt}}\,x \equiv f_{\mathsf{tt}}\,y \mathrel{\dot\to} x \equiv y \\ \dot\forall xy.\, f_{\mathsf{ff}}\,x \not\equiv \star \mathrel{\dot\to} f_{\mathsf{ff}}\,x \equiv f_{\mathsf{ff}}\,y \mathrel{\dot\to} x \equiv y \\ \dot\forall xy.\, f_{\mathsf{tt}}\,x \equiv f_{\mathsf{ff}}\,y \mathrel{\dot\to} f_{\mathsf{tt}}\,x \equiv \star \mathrel{\dot\land} f_{\mathsf{ff}}\,y \equiv \star \end{pmatrix}
$$

Trakhtenbrot's Theorem

Given an instance R of PCP, we construct a formula φ_R by:

$$\varphi_R := \varphi_P \mathbin{\dot\wedge} \varphi_\prec \mathbin{\dot\wedge} \varphi_f \mathbin{\dot\wedge} \varphi_\triangleright \mathbin{\dot\wedge} \dot\exists x.\, P\,x\,x$$

Crucially, we enforce that P satisfies the inversion principle of $R \triangleright (s, t)$:

$$\varphi_\triangleright := \dot\forall xy.\, P\,x\,y \mathbin{\dot\to} \dot\bigvee_{(s,t)\in R} \mathbin{\dot\vee} \begin{cases} x \equiv \overline{s} \mathbin{\dot\wedge} y \equiv \overline{t} \\ \dot\exists uv.\, P\,u\,v \mathbin{\dot\wedge} x \equiv \overline{s}u \mathbin{\dot\wedge} y \equiv \overline{t}v \mathbin{\dot\wedge} u/v \prec x/y \end{cases}$$

Theorem

PCP R *iff* FSATEQ$(\Sigma_{\mathrm{BPCP}}; \equiv)\varphi_R$, *hence* PCP $\preccurlyeq$ FSATEQ$(\Sigma_{\mathrm{BPCP}}; \equiv)$.

Proof.

If R has a solution of length n, then φ_R is satisfied by the model of strings of length bounded by n. Conversely, if $\mathcal{M} \vDash_\rho \varphi_R$ we can extract a solution of R from $\varphi_\triangleright$ by well-founded induction on $\prec^{\mathcal{M}}$ (which is applicable since $\mathcal{M}$ is finite). □

Signature Transformations

Given a finite and discrete signature Σ with arities bounded by n, we have:

$$\mathsf{FSATEQ}(\Sigma; \equiv) \preccurlyeq \mathsf{FSAT}(\Sigma) \preccurlyeq \mathsf{FSAT}(\mathbb{0}; P^{n+2}) \preccurlyeq \mathsf{FSAT}(\mathbb{0}; \in^2)$$

First reduction: axiomatise that $\equiv$ is a congruence for the symbols in Σ

Second reduction:

- Encode k-ary functions as $(k+1)$-ary relations
- Align the relation arities to be constantly $n+1$
- Merge relations into a single $(n+2)$-ary relation indexed by constants
- Interpret constants with fresh variables

Caveat: intermediate reductions may rely on discrete models...

Discrete Models

$\mathsf{FSAT}'(\Sigma)\,\varphi$ if $\mathsf{FSAT}(\Sigma)\,\varphi$ on a discrete model

Can every finite model $\mathcal{M}$ be transformed to a discrete finite model $\mathcal{M}'$?

Idea: first-order indistinguishability $x \doteq y \;:=\; \forall \varphi \rho.\, \mathcal{M} \vDash_{x\cdot\rho} \varphi \leftrightarrow \mathcal{M} \vDash_{y\cdot\rho} \varphi$

Lemma

The relation $x \doteq y$ is a decidable congruence for the symbols in Σ.

Fact

$\mathsf{FSAT}'(\Sigma)\,\varphi$ *iff* $\mathsf{FSAT}(\Sigma)\,\varphi$, *hence in particular* $\mathsf{FSAT}'(\Sigma)\,\varphi \preccurlyeq \mathsf{FSAT}(\Sigma)\,\varphi$.

Proof.

If $\mathcal{M} \vDash_\rho \varphi$ pick $\mathcal{M}'$ to be the quotient of $\mathcal{M}$ under $x \doteq y$. $\square$

Compressing Relations: $\mathsf{FSAT}(\mathbb{0}; P^n) \preccurlyeq \mathsf{FSAT}(\mathbb{0}; \in^2)$

Intuition: encode $P\,x_1 \dots x_n$ as $(x_1, \dots, x_n) \in p$ for a set p representing P

So let's play set theory! For a set d representing the domain we define $\varphi'_\in$:

$$\begin{array}{rclrcl} (P\,x_1 \dots x_n)'_\in & := & \text{“}(x_1, \dots, x_n) \in p\text{”} & (\dot\forall z.\,\varphi)'_\in & := & \dot\forall z.\, z \in d \mathrel{\dot\to} (\varphi)'_\in \\ (\varphi \mathbin{\dot\Box} \psi)'_\in & := & (\varphi)'_\in \mathbin{\dot\Box} (\psi)'_\in & (\dot\exists z.\,\varphi)'_\in & := & \dot\exists z.\, z \in d \mathbin{\dot\wedge} (\varphi)'_\in \end{array}$$

Then $\varphi_\in$ is $\varphi'_\in$ plus asserting $\in$ to be extensional and d to be non-empty.

Fact

$\mathsf{FSAT}(\mathbb{0}; P^n)\,\varphi$ *iff* $\mathsf{FSAT}(\mathbb{0}; \in^2)\,\varphi_\in$, *hence* $\mathsf{FSAT}(\mathbb{0}; P^n) \preccurlyeq \mathsf{FSAT}(\mathbb{0}; \in^2)$.

Proof.

The hard direction is to construct a model of $\varphi_\in$ given a model $\mathcal{M}$ of φ. We employ a segment of the model of hereditarily finite sets by Smolka and Stark (2016) large enough to accommodate $\mathcal{M}$. $\square$

Full Signature Classification

Composing all signature transformations verified we obtain:

Theorem

If Σ contains either an at least binary relation or a unary relation together with an at least binary function, then PCP *reduces to* $\mathsf{FSAT}(\Sigma)$.

On the other hand, FSAT for monadic signatures remains decidable:

Theorem

If Σ is discrete and has all arities bounded by 1 *or if all relation symbols have arity* 0*, then* $\mathsf{FSAT}(\Sigma)$ *is decidable.*

In any case, since one can enumerate all finite models up to extensionality:

Fact

If Σ is discrete and enumerable, then $\mathsf{FSAT}(\Sigma)$ *is enumerable.*

Example 3: First-Order Axiom Systems*

*K. and Marc Hermes at ITP'21.

General Idea

Is a formula φ entailed by an axiomatisation A?

Strategy if A is strong enough to capture computation:

- Encode Turing machine M as formula φ_M
- Verify that M halts iff $A \vDash \varphi_M$
- Verify that M halts iff $A \vdash \varphi_M$ ($\rightarrow$ direction by hand)
- Instead of TM use problems suitable to encode in A

As hard as consistency and incompleteness:

- Reducing a non-trivial problem P to $A \vdash \varphi$ shows A consistent
- Undecidability implies incompleteness for enumerable axiomatisations

Connections to Consistency and Incompleteness

Fact (Consistency)

If $p \preceq A^{\vdash}$ and there is x with $\neg p\, x$ then $A \nvdash \bot$.

Proof.

Let f witness $p \preceq A^{\vdash}$. Then $A \nvdash f\, x$ by $\neg p\, x$ and thus $A \nvdash \bot$. □

Fact (Synthetic Incompleteness)

If A is complete ($\forall \varphi.\, A \vdash \varphi \lor A \vdash \neg\varphi$) and consistent, then $A^{\vdash}$ is decidable.

Proof.

By application of Post's theorem. The premises are enumerability of $A^{\vdash}$ (immediate), enumerability of its complement (as $A \nvdash \varphi$ iff $A \vdash \neg\varphi$), and logical decidability of $A^{\vdash}$ (as $A \vdash \varphi \lor A \vdash \neg\varphi$ implies $A \vdash \varphi \lor A \nvdash \varphi$). □

Sketch for Peano Arithmetic

Use axiomatisation PA over standard signature $(0, \mathsf{S}, +, \cdot\,; \equiv)$.

Diophantine constraints (cf. Larchey-Wendling and Forster (2019)):

- Instances are lists L of constraints $x_i = 1 \mid x_i + x_j = x_k \mid x_i \cdot x_j = x_k$
- L is solvable if there is an evaluation $\eta : \mathbb{N} \to \mathbb{N}$ solving all constraints

Theorem

$L = [c_1, \dots, c_k]$ with maximal index x_n is solvable iff $\mathsf{PA} \vDash \exists^n c_1 \wedge \dots \wedge c_k$.

Proof.

If L has solution η instantiate the existential quantifiers with numerals $\overline{\eta_1}, \dots, \overline{\eta_n}$. Then the axioms of PA entail the constraints.
If $\mathsf{PA} \vDash \exists^n c_1 \wedge \dots \wedge c_k$ use the standard model $\mathbb{N}$ to extract solution η. $\square$

Fact

$L = [c_1, \dots, c_k]$ with maximal index x_n is solvable iff $\mathsf{PA} \vdash \exists^n c_1 \wedge \dots \wedge c_k$.

Interlude: Models of ZF

Sets-as-trees interpretation (Aczel (1978)):

- Type $\mathcal{T}$ of well-founded trees with constructor $\tau : \forall X.\, (X \to \mathcal{T}) \to \mathcal{T}$
- Equality of trees s, t given by isomorphism $s \approx t$
- Membership defined by $s \in \tau\, X\, f := \exists x.\, s \approx f\, x$
- Set operations implemented by tree operations:
  - $\emptyset := \tau \perp \mathsf{elim}_\perp$
  - $\{s, t\} := \tau\, \mathbb{B}\, (\lambda b.\, \text{if } b \text{ then } s \text{ else } t)$
  - $\omega := \tau\, \mathbb{N}\, (\lambda n.\, \overline{n})$ where $\overline{0} := \emptyset$ and $\overline{\mathsf{S}\, n} := \overline{n} \cup \{\overline{n}\}$
  - ...

Axioms needed in Coq:

- EM to really interpret ZF instead of IZF
- Replacement needs a type-theoretical choice axiom (Werner (1997))
- Strong quotient axiom for $(\mathcal{T}, \approx)$ suffices (Kirst and Smolka (2019))
- This yields a well-behaved model $\mathcal{S}$: quotiented, standard numbers

Sketch for ZF Set Theory

Use axiomatisation ZF over explicit signature $(\emptyset, \{\_,\_\}, \bigcup, \mathcal{P}, \omega\,; \equiv, \in)$.

Reduction from PCP:

- Boolean encoding: $\overline{\mathrm{tt}} = \{\emptyset\}$ and $\overline{\mathrm{ff}} = \emptyset$
- String encoding: $\overline{\mathrm{tt\,ff\,ff\,tt}} = (\overline{\mathrm{tt}}, (\overline{\mathrm{ff}}, (\overline{\mathrm{tt}}, (\overline{\mathrm{ff}}, \emptyset))))$
- Stack encoding: $\overline{S} = \{(\overline{s_1}, \overline{t_1}), \dots, (\overline{s_k}, \overline{t_k})\}$
- Combination encoding: $S +\!\!+ B := \bigcup_{s/t \in S} \{(\overline{s}x, \overline{t}y) \mid (x,y) \in B\}$
- $f \triangleright n := (\emptyset, \overline{S}) \in f \land \forall (k, B) \in f.\, k \in n \rightarrow (k+1, S +\!\!+ B) \in f$

$$\varphi_S := \exists f, n, B, x.\, n \in \omega \land f \triangleright n \land (n, B) \in f \land (x, x) \in B$$

Theorem

PCP S *iff* ZF $\vDash \varphi_S$ *and* PCP S *iff* ZF $\vdash \varphi_S$.

Proof.

Direction $\rightarrow$ by proofs in ZF and $\leftarrow$ relies on standard model $\mathcal{S}$. $\square$

Conclusion

Ongoing and Future Work

- Undecidability and incompleteness of finitary set theories
- Minimalistic undecidability proof for the binary signature
- Undecidability and incompleteness of second-order logic
- Constructive analysis of Tennenbaum's theorem
- Engineering: tool support, connect Coq developments

Take-Home Messages

- Synthetic computability: elegant formalism, feasible to mechanise
- Metamathematics: rewarding to revisit in constructive type theory
- Coq mechanisation: implements constructive proofs as algorithms
- If you work on undecidability proofs in Coq:
  Our library could help you and is open for contributions

Thank You!

Bibliography I

Aczel, P. (1978). The type theoretic interpretation of constructive set theory. In *Studies in Logic and the Foundations of Mathematics*, volume 96, pages 55–66. Elsevier.

Bauer, A. (2006). First steps in synthetic computability theory. *Electronic Notes in Theoretical Computer Science*, 155:5 – 31. Proceedings of the 21st Annual Conference on Mathematical Foundations of Programming Semantics (MFPS XXI).

Börger, E., Grädel, E., and Gurevich, Y. (1997). *The Classical Decision Problem*. Perspectives in Mathematical Logic. Springer-Verlag Berlin Heidelberg.

Coquand, T. and Huet, G. (1986). *The calculus of constructions*. PhD thesis, INRIA.

Forster, Y., Kirst, D., and Smolka, G. (2019). On synthetic undecidability in coq, with an application to the entscheidungsproblem. In *Proceedings of the 8th ACM SIGPLAN International Conference on Certified Programs and Proofs*.

Forster, Y., Kirst, D., and Wehr, D. (2020a). Completeness Theorems for First-Order Logic Analysed in Constructive Type Theory. In *Symposium on Logical Foundations Of Computer Science, 2020, Deerfield Beach, Florida, U.S.A.*

Forster, Y., Larchey-Wendling, D., Dudenhefner, A., Heiter, E., Kirst, D., Kunze, F., Smolka, G., Spies, S., Wehr, D., and Wuttke, M. (2020b). A Coq Library of Undecidable Problems. In *CoqPL 2020*, New Orleans, LA, United States.

Bibliography II

Gödel, K. (1931). Über formal unentscheidbare sätze der principia mathematica und verwandter systeme i. *Monatshefte für mathematik und physik*, 38(1):173–198.

Gödel, K. (1930). Die Vollständigkeit der Axiome des logischen Funktionenkalküls. *Monatshefte für Mathematik und Physik*, 37:349–360.

Kirst, D. and Hermes, M. (2021). Synthetic undecidability and incompleteness of first-order axiom systems in coq. In *12th International Conference on Interactive Theorem Proving (ITP 2021)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik.

Kirst, D. and Larchey-Wendling, D. (2020). Trakhtenbrot's theorem in coq, a constructive approach to finite model theory. *arXiv preprint arXiv:2004.07390*.

Kirst, D. and Smolka, G. (2019). Categoricity results and large model constructions for second-order zf in dependent type theory. *Journal of Automated Reasoning*, 63(2):415–438.

Larchey-Wendling, D. and Forster, Y. (2019). Hilbert's Tenth Problem in Coq. In *4th International Conference on Formal Structures for Computation and Deduction*, volume 131 of *LIPIcs*, pages 27:1–27:20.

Libkin, L. (2010). *Elements of Finite Model Theory*. Springer Publishing Company, Incorporated, 1st edition.

Manna, Z. (2003). *Mathematical theory of computation*. Dover Publications, Inc.

Bibliography III

Paulin-Mohring, C. (1993). Inductive definitions in the system coq rules and properties. In *International Conference on Typed Lambda Calculi and Applications*, pages 328–345. Springer.

Richman, F. (1983). Church's thesis without tears. *The Journal of symbolic logic*, 48(3):797–803.

Smolka, G. and Stark, K. (2016). Hereditarily Finite Sets in Constructive Type Theory. In *Interactive Theorem Proving - 7th International Conference, ITP 2016, Nancy, France, August 22-27, 2016*, volume 9807 of *LNCS*, pages 374–390. Springer.

Tarski, A. (1953). I: A general method in proofs of undecidability. In Tarski, A., editor, *Undecidable Theories*, volume 13 of *Studies in Logic and the Foundations of Mathematics*, pages 1–34. Elsevier.

The Coq Development Team (2021). The coq proof assistant.

Werner, B. (1997). Sets in types, types in sets. In *International Symposium on Theoretical Aspects of Computer Software*, pages 530–546. Springer.