

Towards Extraction of Continuity Moduli in Coq

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Background

- Florian works on a Coq library of computable analysis¹
 - ▶ Typical goal: prove some particular $f : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$ continuous
 - ▶ Doing this by hand is repetitive and in fact unnecessary, since...
 - ▶ Metatheorem: in vanilla Coq, every definable function is continuous
- Yannick works on the MetaCoq project²
 - ▶ Internal representation of Coq terms with quote and unquote
 - ▶ Support for monadic programming to implement metatheorems
- I learned about the continuity theorem for System T at PC'18
 - ▶ Translated Escardó's Agda implementation³ to Coq

Project idea when we met at PC'19:
Implement a PoC continuity plugin

¹[Steinberg et al. '19] ²[Sozeau et al. '19] ³[Escardó '13]

Demo

```
(** * Demo *)  
  
Require Import systemT.  
  
(* A simple first example *)  
  
Definition f (a : N → N) := a 7.  
  
Goal continuous f.  
Proof.  
  MetaCoq Run (ExtractModulus "cont_f" f).  
  Check cont_f.  
  exact cont_f.  
  
(* A failing example *)  
  
Definition f2 (a : N → N) := a (a 3 + a 4).  
  
Fail MetaCoq Run (ExtractModulus "cont_f2" f2).  
  
Fail MetaCoq Run (Reify "ext_add" Nat.add).  
  
(* A more careful example *)
```

```
0:00 *goal* 0:00 (Coq Goal)  
0:00 *response* 0:00 (Coq Response)  
  
0:00 *goal* 0:00 (Coq Goal)  
0:00 *response* 0:00 (Coq Response)
```

Demo

```
(** * Demo *)  
  
Require Import systemT.  
  
(* A simple first example *)  
  
Definition f (a : N → N) := a 7.  
  
Goal continuous f.  
Proof.  
  MetaCoq Run (ExtractModulus "cont_f" f).  
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(* A failing example *)  
  
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Fail MetaCoq Run (ExtractModulus "cont_f2" f2).  
  
Fail MetaCoq Run (Reify "ext_add" Nat.add).  
  
(* A more careful example *)
```

```
[100% *quals*] [111 (1,0)] [Coq moduli]  
f is defined  
  
[100% *response*] [111 (1,0)] [Coq Response Moduli]  
Top (0,32) Bitmonitor (Coq Script(B)) company-▶ yes to Get! company names
```

Demo

The screenshot shows a CoqIDE session window titled "demo.v". The left pane displays the Coq code being developed:

```
(** * Demo *)
Require Import systemT.

(* A simple first example *)

Definition f (a : N → N) := a 7.

Goal continuous f.
Proof.
  MetaCoq Run (ExtractModulus "cont_f" f).
  Check cont_f.
  exact cont_f.
  
```

The right pane shows the state of the proof development:

1 subgoal (ID 286)

continuous f

At the bottom, there are two tabs: "goals" and "responses".

Goals tab (left):

- 1 subgoal (ID 286)
- continuous f

Responses tab (right):

- 1 subgoal (ID 286)
- continuous f

Bottom status bar:

demo.v Top (H,10) Bitmarker (Coq Script(1-> company-● yes to Get! company names))

demo.v Top (H,10) Bitmarker (Coq Response (no)-bitmark)

Demo

The screenshot shows a CoqIDE session window titled "demo.v". The left pane displays the Coq code, and the right pane shows the proof state and responses from the MetaCoq server.

Code (Left Pane):

```
(** * Demo *)  
Require Import systemT.  
  
(* A simple first example *)  
  
Definition f (a : N → N) := a 7.  
  
Goal continuous f.  
Proof.  
  MetaCoq Run (ExtractModulus "cont_f" f).  
  Check cont_f.  
  exact cont_f.  
  
(* A failing example *)  
  
Definition f2 (a : N → N) := a (a 3 + a 4).  
  
Fail MetaCoq Run (ExtractModulus "cont_f2" f2).  
  
Fail MetaCoq Run (Reify "ext_add" Nat.add).  
  
(* A more careful example *)
```

Proof State (Right Pane):

```
1 subgoal (ID 286)  
  
continuous f
```

MetaCoq Server Responses (Bottom):

```
[1] 00 - *goals*          [1] 00 - (L1, R1)          [1] 00 - (Log (modf_a))  
[1] 00 - *response*      [1] 00 - (L1, R1)          [1] 00 - (Log Response Modf-a)
```

Demo

The screenshot shows a Coq development environment with two panes. The left pane displays a script named `demo.v` containing Coq code. The right pane shows the state of the proof assistant.

`demo.v` content:

```
(** * Demo *)  
Require Import systemT.  
  
(* A simple first example *)  
  
Definition f (a : N → N) := a 7.  
  
Goal continuous f.  
Proof.  
  MetaCoq Run (ExtractModulus "cont_f" f).  
  Check cont_f.  
  exact cont_f.  
  
(* A failing example *)  
  
Definition f2 (a : N → N) := a (a 3 + a 4).  
  
Fail MetaCoq Run (ExtractModulus "cont_f2" f2).  
  
Fail MetaCoq Run (Reify "ext_add" Nat.add).  
  
(* A more careful example *)
```

Right pane status bar:

```
demo.v Top (13,15) Bitmarker (Coq Script(1-) company-● yes to Get! company names)  
demo.v 1 subgoal (ID 286)  
continuous f  
cont_f : continuous f  
Top (13,15) Bitmarker (Coq Response(1-) company-● yes to Get! company names)
```

Demo

```
(** * Demo *)  
  
Require Import systemT.  
  
(* A simple first example *)  
  
Definition f (a : N → N) := a 7.  
  
Goal continuous f.  
Proof.  
  MetaCoq Run (ExtractModulus "cont_f" f).  
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(* A failing example *)  
  
Definition f2 (a : N → N) := a (a 3 + a 4).  
  
Fail MetaCoq Run (ExtractModulus "cont_f2" f2).  
  
Fail MetaCoq Run (Reify "ext_add" Nat.add).  
  
(* A more careful example *)
```

Demo

The screenshot shows a Coq development environment with two panes. The left pane contains the source code in `Demo.v`, and the right pane shows the results of the code execution.

`Demo.v` content:

```
(** * Demo *)  
Require Import systemT.  
  
(* A simple first example *)  
  
Definition f (a : N → N) := a 7.  
  
Goal continuous f.  
Proof.  
  MetaCoq Run (ExtractModulus "cont_f" f).  
  Check cont_f.  
  exact cont_f.  
  
(* A failing example *)  
  
Definition f2 (a : N → N) := a (a 3 + a 4).  
  
Fail MetaCoq Run (ExtractModulus "cont_f2" f2).  
  
Fail MetaCoq Run (Reify "ext_add" Nat.add).  
  
(* A more careful example *)
```

Right pane output:

```
(*goal*) 0(1)(1,0) (Coq.Geo.C)  
Unnamed_thm is defined
```

Demo

```
demo.v 1:8 system.v 2:10 demo.v
Check cont_f.
exact cont_f.

(* A failing example *)

Definition f2 (a : N → N) := a (a 3 + a 4).|  
Fail MetaCoq Run (ExtractModulus "cont_f2" f2).

Fail MetaCoq Run (Reify "ext_add" Nat.add).

(* A more careful example *)

Definition add := (λ n m : N ⇒ nat_rec _ m (λ n r ⇒ S r) n).

MetaCoq Run (Reify "ext_add" add).

Definition f3 (a : N → N) := a (add (a 3) (a 4)).  
MetaCoq Run (Reify "ext_f3" f3).

MetaCoq Run (ExtractModulus "cont_f3" f3).
```

```
demo.v 1:8 system.v 2:10 demo.v
Check cont_f.
exact cont_f.

(* A failing example *)

Definition f2 (a : N → N) := a (a 3 + a 4).|  
Fail MetaCoq Run (ExtractModulus "cont_f2" f2).

Fail MetaCoq Run (Reify "ext_add" Nat.add).

(* A more careful example *)

Definition add := (λ n m : N ⇒ nat_rec _ m (λ n r ⇒ S r) n).

MetaCoq Run (Reify "ext_add" add).

Definition f3 (a : N → N) := a (add (a 3) (a 4)).|  
f2 is defined  
MetaCoq Run (Reify "ext_f3" f3).

MetaCoq Run (ExtractModulus "cont_f3" f3).

Check cont_f.
```

Demo

```
demo.v      SystemT
Check cont_f.
exact cont_f.

(* A failing example *)

Definition f2 (a : N → N) := a (a 3 + a 4).

Fail MetaCoq Run (ExtractModulus "cont_f2" f2).

Fail MetaCoq Run (Reify "ext_add" Nat.add).

(* A more careful example *)

Definition add := (λ n m : N ⇒ nat_rec _ m (λ n r ⇒ S r) n).

MetaCoq Run (Reify "ext_add" add).

Definition f3 (a : N → N) := a (add (a 3) (a 4)).

MetaCoq Run (Reify "ext_f3" f3).

MetaCoq Run (ExtractModulus "cont_f3" f3).
```

The command has indeed failed with message:
not in SystemT fragment

```
demo.v      SystemT
Check cont_f.
```

```
demo.v      SystemT
Check cont_f.
```

Demo

```
demo.v      SystemT
Check cont_f.
exact cont_f.

(* A failing example *)

Definition f2 (a : N → N) := a (a 3 + a 4).

Fail MetaCoq Run (ExtractModulus "cont_f2" f2).

Fail MetaCoq Run (Reify "ext_add" Nat.add).

(* A more careful example *)

Definition add := (λ n m : N ⇒ nat_rec _ m (λ n r ⇒ S r) n).

MetaCoq Run (Reify "ext_add" add).

Definition f3 (a : N → N) := a (add (a 3) (a 4)).

MetaCoq Run (Reify "ext_f3" f3).

MetaCoq Run (ExtractModulus "cont_f3" f3).
```

The command has indeed failed with message:
not in SystemT fragment

Check cont_f.

demo.v SystemT
Check cont_f.

Demo

```
Fail MetaCoq Run (Reify "ext_add" Nat.add).
```

```
(* A more careful example *)
```

```
Definition add := ( $\lambda n m : \mathbb{N} \Rightarrow \text{nat\_rec } m (\lambda n r \Rightarrow S r) n$ ).
```

```
MetaCoq Run (Reify "ext_add" add).
```

```
Definition f3 (a :  $\mathbb{N} \rightarrow \mathbb{N}$ ) := a (add (a 3) (a 4)).
```

```
MetaCoq Run (Reify "ext_f3" f3).
```

```
MetaCoq Run (ExtractModulus "cont_f3" f3).
```

```
Check cont_f3.
```

```
(* Modulus extraction *)
```

```
Print continuous.
```

```
Compute (get_modulus cont_f3 ( $\lambda n \Rightarrow 3$ )).
```

```
Compute (get_modulus cont_f3 ( $\lambda n \Rightarrow n$ )).
```

```
add is defined
```

--- 880.v Ret (29,60) Bitmarker (Coq Script(0-2) company-● yes to Gett company files)

--- 880.*response* Ret (1,4) (Coq Response Web-Interface)

Demo

```
Fail MetaCoq Run (Reify "ext_add" Nat.add).

(* A more careful example *)

Definition add := ( $\lambda n m : \mathbb{N} \Rightarrow \text{nat\_rec } _- m (\lambda n r \Rightarrow S r) n$ ).

MetaCoq Run (Reify "ext_add" add).

Definition f3 (a :  $\mathbb{N} \rightarrow \mathbb{N}$ ) := a (add (a 3) (a 4)).

MetaCoq Run (Reify "ext_f3" f3).

MetaCoq Run (ExtractModulus "cont_f3" f3).

Check cont_f3.

(* Modulus extraction *)

Print continuous.

Compute (get_modulus cont_f3 ( $\lambda n \Rightarrow 3$ )).

Compute (get_modulus cont_f3 ( $\lambda n \Rightarrow n$ )).
```

```
(ilambda (ilambda (iapp (iapp (iapp (irec N) (ivar F1))  
                  (ilambda (ilambda (iapp isucc (ivar F1)))))) (ivar (FS F1))))
```

Demo

```
Fail MetaCoq Run (Reify "ext_add" Nat.add).

(* A more careful example *)

Definition add := ( $\lambda$  n m : N  $\Rightarrow$  nat_rec _ m ( $\lambda$  n r  $\Rightarrow$  S r) n).

MetaCoq Run (Reify "ext_add" add).

Definition f3 (a : N  $\rightarrow$  N) := a (add (a 3) (a 4)).
```

MetaCoq Run (Reify "ext_f3" f3).

MetaCoq Run (ExtractModulus "cont_f3" f3).

Check cont_f3.

(* Modulus extraction *)

Print continuous.

Compute (get_modulus cont_f3 (λ n \Rightarrow 3)).

Compute (get_modulus cont_f3 (λ n \Rightarrow n)).

f₃ is defined

Demo

```
Fail MetaCoq Run (Reify "ext_add" Nat.add).  
  
(* A more careful example *)  
  
Definition add := ( $\lambda n m : \mathbb{N} \Rightarrow \text{nat\_rec}_- m (\lambda n r \Rightarrow S r) n$ ).  
  
MetaCoq Run (Reify "ext_add" add).  
  
Definition f3 (a :  $\mathbb{N} \rightarrow \mathbb{N}$ ) := a (add (a 3) (a 4)).  
  
MetaCoq Run (Reify "ext_f3" f3).  
  
MetaCoq Run (ExtractModulus "cont_f3" f3).  
  
Check cont_f3.  
  
(* Modulus extraction *)  
  
Print continuous.  
  
Compute (get_modulus cont_f3 ( $\lambda n \Rightarrow 3$ )).  
  
Compute (get_modulus cont_f3 ( $\lambda n \Rightarrow n$ )).
```

```
UML - *goal*          all (1,0)      (Obj. goal*)  
(ilambda  
  (iapp (ivar F1)  
    (iapp  
      (iapp (ilambda (ilambda (iapp (iapp (iapp (irec N)  
        (ivar F1)) (ilambda (ilambda (iapp isucc (ivar F1)))) (ivar  
        (FS F1))))))  
      (iapp (ivar F1) (iapp isucc (iapp isucc (iapp  
        isucc izero)))) (iapp (ivar F1) (iapp isucc (iapp isucc  
        (iapp isucc (iapp isucc izero)))))))
```

Demo

```
Fail MetaCoq Run (Reify "ext_add" Nat.add).
```

```
(* A more careful example *)
```

```
Definition add := ( $\lambda$  n m : N  $\Rightarrow$  nat_rec _ m ( $\lambda$  n r  $\Rightarrow$  S r) n).
```

```
MetaCoq Run (Reify "ext_add" add).
```

```
Definition f3 (a : N  $\rightarrow$  N) := a (add (a 3) (a 4)).
```

```
MetaCoq Run (Reify "ext_f3" f3).
```

```
MetaCoq Run (ExtractModulus "cont_f3" f3). |
```

```
Check cont_f3.
```

```
(* Modulus extraction *)
```

```
Print continuous.
```

```
Compute (get_modulus cont_f3 ( $\lambda$  n  $\Rightarrow$  3)).
```

```
Compute (get_modulus cont_f3 ( $\lambda$  n  $\Rightarrow$  n)).
```

Ret (37,40) Bitmoner (Coq Script(0-2) company-● yes to Gett company files)

0:00 *goal* 0:00 (1,4) (Coq Modulus)

0:00 *response* 0:00 (1,4) (Coq Response Modulus)

Demo

The screenshot shows a terminal window with two panes. The left pane displays a sequence of MetaCoq commands and their responses. The right pane shows the CoqIDE interface with a proof state and a response buffer.

```
Fail MetaCoq Run (Reify "ext_add" Nat.add).  
  
(* A more careful example *)  
  
Definition add := ( $\lambda n m : \mathbb{N} \Rightarrow \text{nat\_rec } \_ m (\lambda n r \Rightarrow S r) n$ ).  
  
MetaCoq Run (Reify "ext_add" add).  
  
Definition f3 (a :  $\mathbb{N} \rightarrow \mathbb{N}$ ) := a (add (a 3) (a 4)).  
  
MetaCoq Run (Reify "ext_f3" f3).  
  
MetaCoq Run (ExtractModulus "cont_f3" f3).  
  
Check cont_f3.  
  
(* Modulus extraction *)  
  
Print continuous.  
  
Compute (get_modulus cont_f3 ( $\lambda n \Rightarrow 3$ )).  
  
Compute (get_modulus cont_f3 ( $\lambda n \Rightarrow n$ )).
```

Right pane details:

- Top status bar: 0:00 *goals* 0:00 (1,0) (Coq Moduli)
- Proof state:
cont_f3
: continuous f3
- Bottom status bar: 0:00 *response* 0:00 (1,0) (Coq Response Module)

Demo

Check cont_f3.

(* Modulus extraction *)

Print continuous.

Compute (get_modulus cont_f3 ($\lambda n \Rightarrow 3$)).

Compute (get_modulus cont_f3 ($\lambda n \Rightarrow n$)).

```
continuous =  $\lambda (X : \text{Type}) (f : \text{Baire} \rightarrow X) \Rightarrow \forall a : N \rightarrow$ 
 $N, \{L : \text{list } N \mid \forall \beta : N \rightarrow N, a = [L] \beta \rightarrow f a = f \beta\}$ 
 $: \forall X : \text{Type}, (\text{Baire} \rightarrow X) \rightarrow \text{Set}$ 
```

Argument X is implicit

Argument scopes are [type_scope function_scope]

Demo

The screenshot shows a Coq development environment with two panes. The left pane is titled "demo.v" and contains the following code:

```
Check cont_f3.  
  
(* Modulus extraction *)  
  
Print continuous.  
  
Compute (get_modulus cont_f3 ( $\lambda$  n  $\Rightarrow$  3)).  
  
Compute (get_modulus cont_f3 ( $\lambda$  n  $\Rightarrow$  n)).
```

The right pane is titled "response" and displays the results of the computations:

```
[[00- *goal*] : (1) (1,0) ((Coq.Moduli))  
= [3; 4; 6]  
: list N
```

At the bottom of the interface, there are tabs for "demo.v" and "response".

Demo

The screenshot shows a Coq development environment with two panes. The left pane contains a proof script, and the right pane shows the resulting response.

Left Pane (Script):

```
Check cont_f3.  
  
(* Modulus extraction *)  
  
Print continuous.  
  
Compute (get_modulus cont_f3 ( $\lambda$  n  $\Rightarrow$  3)).  
  
Compute (get_modulus cont_f3 ( $\lambda$  n  $\Rightarrow$  n)).
```

Right Pane (Response):

```
U.00 : *{goal}*          A1 : (A,B)          (Coq) (modz)  
= [3; 4; 7]  
: list N
```

Plugin Pipeline

Expected input: T-definable functional $f : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$

- 1 Reify f to the internal MetaCoq representation f_C
- 2 Try to translate f_C to an untyped T-term f_U
- 3 Infer a type for f_U to obtain an intrinsically typed T-term f_T
- 4 Compute the continuity information for f_T
- 5 Verify that f_T corresponds to f

Gödel's System T

Simply typed lambda calculus with natural numbers and recursors:

$$A, B ::= \mathbb{N} \mid A \rightarrow B$$

$$s, t ::= x \mid \lambda x. s \mid s t \mid 0 \mid S \mid R_A$$

Usual typing rules $\Gamma \vdash s : A$ of simply typed lambda calculus plus:

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \frac{}{\Gamma \vdash S : \mathbb{N} \rightarrow \mathbb{N}} \quad \frac{}{\Gamma \vdash R_A : A \rightarrow (\mathbb{N} \rightarrow A \rightarrow A) \rightarrow \mathbb{N} \rightarrow A}$$

Natural denotational semantics in type theory:
Judgements $\vdash s : A$ translate to terms $\llbracket s \rrbracket : \llbracket A \rrbracket$

Continuity of T-Definable Functionals (cf. Escardó '13)

A functional $f : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$ is continuous if

- “it accesses only finitely many positions of every input sequence”
- $\forall(\alpha : \mathbb{N} \rightarrow \mathbb{N}). \Sigma(L : \mathbb{N}^*). \forall(\beta : \mathbb{N} \rightarrow \mathbb{N}). \alpha \approx_L \beta \rightarrow f\alpha = f\beta$ ¹

¹Projecting out $\mu_f : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}^*$ yields the **modulus of continuity**.

Theorem

If $\vdash s : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$ is typable in System T, then $\llbracket s \rrbracket$ is continuous.

Proof sketch.

We translate Escardó's Agda development to Coq

- Logical statements placed in impredicative universe of propositions
- Computationally relevant notions placed in predicative type universes
- Based on intrinsically typed de Bruijn representation of System T

Reification (cf. Forster/Kunze '19)

- MetaCoq type `Ast.tm` represents untyped Coq terms
- MetaCoq program `tmQuote` reifies Coq terms to `Ast.tm`

```
Module Ast.  
Inductive tm : Set :=  
| tRel : nat -> tm  
| tConstruct : inductive -> nat ->  
    universe_instance -> tm  
| tFix : mfixpoint tm -> nat -> tm  
| tLambda : name -> tm -> tm -> tm  
| tApp : tm -> tm -> tm  
| (* ... *).  
End Ast.
```

```
Module SystemT.  
Inductive tm : Type :=  
| var : nat -> tm  
| zero : tm  
| succ : tm  
| rec : type -> tm  
| lambda : type -> tm -> tm  
| app : tm -> tm -> tm.  
End SystemT.
```

- Implemented translation to `SystemT.tm` as monadic program `reify`

Type Inference

- Explicit annotations in SystemT.tm allow for unique type inference
- Composes to reification from Coq functions f to typed T-terms f_T

```
Definition Reify (def : ident) {A} (f : A) :=  
  f <- tmEval hnf f;;  
  s <- tmQuote f;;  
  s' <- reify 42 (trans s);;  
  s' <- tmEval cbv (infer s' empty_env);;  
  match s' with  
  | Some (_; s') => tmDefinitionRed def (Some Common.hnf) s'  
  | None => tmFail "could not infer type"  
  end.
```

Extraction Plugin

- Feeds the extracted typed T-term f_T into the continuity theorem
- Tries to show that $f = \llbracket f_T \rrbracket$ by reflexivity
- Concludes the continuity of f

```
Definition ExtractModulus def f :=
  syn <- tmQuote f;;
  m <- ExtractModulus' f;;
  let (A, s) := m in
  match type_eq A ((N -> N) -> N) with
  | left H => s <- tmQuote s;;
    H <- tmUnquoteTyped (continuous f) (cnst syn s);;
    H <- tmEval cbv H;;
    tmDefinition def H
  | right _ => @tmFail (continuous f) "wrong type"
end.
```

Debatable Design Decisions

- Syntax translations as monadic programs instead of Coq functions
 - ▶ `Ast.tm` to `SystemT.tm` is morally the identity function
 - ▶ Normalisation needed to eliminate unexpressible features
 - ▶ Supported by MetaCoq's `tmEval` program
 - ▶ Not internally verifiable!
- Ad-hoc type inference for reified terms
 - ▶ Forget typing information of `Ast.tm`, can be easily reconstructed
 - ▶ Allows simple syntax transformations to `SystemT.tm` in empty context
 - ▶ Only works for fully annotated representation of System T!
- Intermediate language to express the supported fragment
 - ▶ Natural starting point given previous developments
 - ▶ Works for well-defined fragments of Coq
 - ▶ Might be difficult to adjust to more language features!

Future Directions

Extend the supported syntax fragment to

- More data types like \mathbb{B} , sums, pairs, lists, rational numbers etc.
- Definitions using Coq's fix/match instead of explicit recursors
- More expressiveness with dependent and informative types
 - ⇒ Clarify how notion and status of continuity scale!

Verify the current pipeline:

- Requires verified normalisation and type inference (Sozeau et al. '20)
- Soundness: $f = \llbracket f_T \rrbracket$ holds whenever reification succeeds
- Completeness: reification succeeds for all T-definable f

<http://www.ps.uni-saarland.de/extras/modulus-extraction/>

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Continuity of T-Definable Functionals (cf. Escardó '13)

Theorem

If $\vdash s : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$ is typable in System T, then $\llbracket s \rrbracket$ is continuous.

Proof sketch.



content