Synthetic Versions of the Kleene-Post and Post's Theorem

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Synthetic Computability Theory[1]

Exploit that in constructive foundations, every definable function is computable:

$$\begin{aligned} A : X \to \mathbb{P} \text{ is decidable} \quad &:= \quad \exists d : X \to \mathbb{B}. \forall x. A\, x \leftrightarrow d\, x = \text{true} \\ A : X \to \mathbb{P} \text{ many-one-reduces to } B : Y \to \mathbb{P} \quad &:= \quad \exists r : X \to Y. \forall x. A\, x \leftrightarrow B\, (r\, x) \end{aligned}$$

Pros:

- Avoid manipulating Turing machines or equivalent model of computation
- Elegant formalisation (e.g. in CIC), feasible mechanisation (e.g. in Coq)

Cons:

- Finding a correct synthetic rendering of Turing reductions not so straightforward
- But Turing reductions are needed for interesting results like Kleene-Post and Post

[1] Richman (1983); Bauer (2006)

Synthetic Oracle Machines

We had some failed attempts, Andrej Bauer's proposal (Bauer (2021)) came to our rescue[2]

A synthetic oracle machine is an operation on functional relations $\mathbb{N} \to \mathbb{B} \to \mathbb{P}$

$$R : \{A : \mathbb{N} \to \mathbb{B} \to \mathbb{P} \mid A \text{ functional}\} \to \{A : \mathbb{N} \to \mathbb{B} \to \mathbb{P} \mid A \text{ functional}\}$$

factoring through a computational core on partial functions $\mathbb{N} \rightharpoonup \mathbb{B}$

$$r : (\mathbb{N} \rightharpoonup \mathbb{B}) \to \mathbb{N} \rightharpoonup \mathbb{B} \qquad \text{with} \qquad \forall f : \mathbb{N} \rightharpoonup \mathbb{B}.\, R\, f = r\, f$$

satisfying the requirement that R be continuous:

$$R\, A\, n\, b \to \exists L : \mathbb{N}^*.\, L \subseteq \mathrm{dom}(A) \wedge \forall A'.\, A' =_L A \to R\, A'\, n\, b$$

Axiom: there is an enumeration R_n of oracle machines

$A \preceq_T B \ := \ A$ Turing-reduces to B if there is an oracle machine R with $R\, B = A$

[2] See Forster (2021) and the related TYPES abstract Forster and Kirst (2022)

The Kleene-Post Theorem (à la Odifreddi (1992))

Goal: construct incomparable Turing degrees $A := \bigcup_{n:\mathbb{N}} \sigma_n$ and $B := \bigcup_{n:\mathbb{N}} \tau_n$

Characterise σ_n and τ_n inductively by a predicate $n \triangleright (\sigma, \tau)$ with:

- If $n \triangleright (\sigma, \tau)$ and $n \triangleright (\sigma', \tau')$, then $\sigma = \sigma'$ and $\tau = \tau'$
- For every n there not not exist σ and τ with $n \triangleright (\sigma, \tau)$
- If $2n \triangleright (\sigma, \tau)$, then $R_n\, A$ differs from B at position $|\tau|$
- If $2n + 1 \triangleright (\sigma, \tau)$, then $R_n\, B$ differs from A at position $|\sigma|$

Theorem (Kleene-Post)

There are predicates A and B such that neither $A \preceq_T B$ nor $B \preceq_T A$.

Post's Theorem (à la Odifreddi (1992))

Goal: connect the arithmetical hierarchy with iterated Turing jumps $\emptyset^{(n)}$

Represent the arithmetical hierarchy on predicates $p : \mathbb{N}^k \to \mathbb{P}$ inductively:

$$\frac{f : \mathbb{N}^k \to \mathbb{B}}{\Sigma_0(\lambda\vec{x}.\, f\,\vec{x} = \mathrm{true})} \qquad \frac{f : \mathbb{N}^k \to \mathbb{B}}{\Pi_0(\lambda\vec{x}.\, f\,\vec{x} = \mathrm{true})} \qquad \frac{\Pi_n\, p}{\Sigma_{n+1}(\lambda\vec{x}.\, \exists y.\, p\,(y :: \vec{x}))} \qquad \frac{\Sigma_n\, p}{\Pi_{n+1}(\lambda\vec{x}.\, \forall y.\, p\,(y :: \vec{x}))}$$

$$\begin{aligned} \text{Turing jump of } A \quad &:= \quad \lambda n.\, R_n\, A\, n\, \mathrm{true} \\ A \text{ is semi-decidable relative to } B \quad &:= \quad \exists R.\, \forall n.\, A\, n \leftrightarrow R\, B\, n\, \mathrm{true} \end{aligned}$$

Theorem (Post)

Assuming LEM ($\forall p.\, p \vee \neg p$), the following can be shown:

- *A predicate A is Σ_{n+1} iff it is semi-decidable relative to $\emptyset^{(n)}$.*
- *If A is Σ_n, then $A \preceq_T \emptyset^{(n)}$. If $n > 0$ already $A \preceq_m \emptyset^{(n)}$ for synthetic many-one reductions.*

Outlook

1. Investigate if the enumeration R_n can be obtained using Church's thesis (Kreisel (1965))
   $\Rightarrow$ Maybe possible using Kleene's second algebra (Kleene (1952))
2. Analyse use of LEM in Post's theorem (though deemed consistent with enumeration R_n)
   $\Rightarrow$ Avoid switching between Σ_n and Π_n via complementation (Akama et al. (2004))
3. Tackle Post's problem regarding an undecidable but enumerable degree below $\emptyset^{(1)}$
   $\Rightarrow$ Following Friedberg (1957) and Mučnik (1956) or Kučera (1986)

Thanks for your attention!

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Backup Kleene-Post

Characterise σ_n and τ_n inductively by $\triangleright : \mathbb{N} \to \mathbb{B}^* \to \mathbb{B}^* \to \mathbb{P}$ with $0 \triangleright (\epsilon, \epsilon)$ and:

$$\frac{2n \triangleright (\sigma, \tau) \quad \sigma' \text{ least extension of } \sigma \text{ with } b = r_n\, \sigma'\, |\tau|}{2n+1 \triangleright (\sigma', \tau +\!\!+ [\neg b])}$$

$$\frac{2n \triangleright (\sigma, \tau) \quad \neg(\exists \sigma' b.\, \sigma' \geq \sigma \land b = r_n\, \sigma'\, |\tau|)}{2n+1 \triangleright (\sigma, \tau +\!\!+ [\mathsf{false}])}$$

$$\frac{2n+1 \triangleright (\sigma, \tau) \quad \tau' \text{ least extension of } \tau \text{ with } b = r_n\, \tau'\, |\sigma|}{2n+2 \triangleright (\sigma +\!\!+ [\neg b], \tau')}$$

$$\frac{2n+1 \triangleright (\sigma, \tau) \quad \neg(\exists \tau' b.\, \tau' \geq \tau \land b = r_n\, \tau'\, |\sigma|)}{2n+2 \triangleright (\sigma +\!\!+ [\mathsf{false}], \tau)}$$

Backup Post

Theorem (Post')

Assuming LEM ($\forall p.\, p \vee \neg p$), the following can be shown:

- *A predicate A is Σ_{n+1} iff it is semi-decidable relative to some B in Π_n.*
- *If A is Σ_n, then $A \preceq_T \emptyset^{(n)}$. If $n > 0$ already $A \preceq_m \emptyset^{(n)}$ for synthetic many-one reductions.*

Lemma

Given an oracle machine R with core r, termination $R\,A\,n\,b$ is equivalent to

$$\exists L_{\mathrm{true}} L_{\mathrm{false}}.\, (\forall n \in L_{\mathrm{true}}.\, A\,b\,\mathrm{true}) \wedge (\forall n \in L_{\mathrm{false}}.\, A\,b\,\mathrm{false}) \wedge r\,(\mathrm{lookup}\,L_{\mathrm{true}}\,L_{\mathrm{false}})\,n = b$$

where lookup $L_{\mathrm{true}}\,L_{\mathrm{false}}\,n$ *returns* true *if* $n \in L_{\mathrm{true}}$, false *if* $n \in L_{\mathrm{false}}$, *and diverges otherwise.*