New Observations on the Constructive Content of First-Order Completeness Theorems

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Analysing Completeness Theorems in Constructive Meta-Theory

Does $\mathcal{T} \vDash \varphi$ imply $\mathcal{T} \vdash \varphi$ constructively?

Confusing situation in the literature on first-order logic:

- Completeness equivalent to Boolean Prime Ideal Theorem (Henkin, 1954)
- Completeness requires Markov's Principle (Kreisel, 1962)
- Completeness equivalent to Weak Kőnig's Lemma (Simpson, 2009)
- Completeness equivalent to Weak Fan Theorem (Krivtsov, 2015)
- Completeness holds fully constructively (Krivine, 1996)

Working Towards an Explanation

There are multiple dimensions at play:

- Syntax fragment (e.g., propositional, minimal, negative, full)
- Complexity of the context (e.g., finite, decidable, enumerable, arbitrary)
- Cardinality of the signature (e.g., countable, uncountable)
- Representation of the semantics (e.g., Boolean, decidable, propositional)

Ongoing systematic investigation using Coq:

- Started by Herbelin and Ilik (2016) and Forster, Kirst, and Wehr (2021)
- New observations by Hagemeier and Kirst (2022) and Kirst (2022)
- Comprehensive overview of current landscape by Herbelin (2022)
- Today: syntactic disjunction, arbitrary contexts, countable signature, prop. semantics

Classical Outline for Intuitionistic Propositional Logic

Employing prime theories ($\varphi \vee \psi \in \mathcal{T} \to \varphi \in \mathcal{T} \vee \varphi \in \mathcal{T}$):

- Lindenbaum Extension: if $\mathcal{T} \nvdash \varphi$ then there is prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash \varphi$
- Universal Model $\mathcal{U}$: consistent prime theories related by inclusion
- Truth Lemma for $\mathcal{T}$ in $\mathcal{U}$: $\varphi \in \mathcal{T} \iff \mathcal{T} \Vdash \varphi$
- Model Existence: if $\mathcal{T} \nvdash \varphi$ then there is $\mathcal{M}$ with $\mathcal{M} \Vdash \mathcal{T}$ and $\mathcal{M} \nVdash \varphi$
- Quasi-Completeness: if $\mathcal{T} \Vdash \varphi$ then $\neg\neg(\mathcal{T} \vdash \varphi)$
- Completeness: if $\mathcal{T} \Vdash \varphi$ then $\mathcal{T} \vdash \varphi$

Constructive Completeness Proof???

For $\mathcal{T}$ quasi-prime $(\varphi \vee \psi \in \mathcal{T} \to \neg\neg(\varphi \in \mathcal{T} \vee \varphi \in \mathcal{T}))$:

- Lindenbaum Extension: if $\mathcal{T} \nVdash \varphi$ then there is quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \nVdash \varphi$
- Universal Model: consistent quasi-prime theories related by inclusion
- Truth Lemma: fails immediately
- Model Existence: fails
- Quasi-Completeness: fails
- Completeness: needs MP/LEM depending on theory complexity and syntax fragment

Constructive Completeness Proof?

For $\mathcal{T}$ quasi-prime ($\varphi \vee \psi \in \mathcal{T} \rightarrow \neg\neg(\varphi \in \mathcal{T} \vee \varphi \in \mathcal{T})$) and stable ($\neg\neg(\varphi \in \mathcal{T}) \rightarrow \varphi \in \mathcal{T}$):

- Lindenbaum Extension: if $\mathcal{T} \nvdash \varphi$ then there is stable quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash \varphi$
- Universal Model: consistent stable quasi-prime theories related by inclusion
- Truth Lemma: fails for disjunction
- Model Existence: fails
- Quasi-Completeness: fails
- Completeness: needs MP/LEM depending on theory complexity and syntax fragment

The Issue with Disjunction

Truth Lemma case for disjunctions $\varphi \vee \psi$:

$$\begin{aligned} \varphi \vee \psi \in \mathcal{T} &\overset{?}{\iff} \mathcal{T} \Vdash \varphi \vee \psi \\ &\overset{def}{\iff} \mathcal{T} \Vdash \varphi \;\vee\; \mathcal{T} \Vdash \psi \\ &\overset{IH}{\iff} \varphi \in \mathcal{T} \;\vee\; \psi \in \mathcal{T} \end{aligned}$$

- So we really need prime theories to interpret disjunctions
- Primeness from Lindenbaum Extension is constructive no-go

Quasi-Completeness via WLEM

Weak law of excluded middle WLEM $:= \forall P : \mathbb{P}. \neg P \vee \neg\neg P$

Lemma

Assuming WLEM, every stable quasi-prime theory is prime.

Proof.

Assume $\varphi \vee \psi \in \mathcal{T}$. Using WLEM, decide whether $\neg(\varphi \in \mathcal{T})$ or $\neg\neg(\varphi \in \mathcal{T})$. In the latter case, conclude $\varphi \in \mathcal{T}$ directly by stability. In the former case, derive $\psi \in \mathcal{T}$ using stability, since assuming $\neg(\psi \in \mathcal{T})$ on top of $\neg(\varphi \in \mathcal{T})$ contradicts quasi-primeness for $\varphi \vee \psi \in \mathcal{T}$. □

Classical proof outline works again up to quasi-completeness!

What happens if we instead weaken the Truth Lemma?

Quasi-Completeness via DNS

Assuming double-negation shift DNS $:= \forall X. \forall p : X \to \mathbb{P}. (\forall x. \neg\neg p\, x) \to \neg\neg(\forall x. p\, x)$:

- Lindenbaum Extension: if $\mathcal{T} \nvdash \varphi$ then there is stable quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash \varphi$
- Universal Model $\mathcal{U}$: consistent stable quasi-prime theories related by inclusion
- Pseudo Truth Lemma for $\mathcal{T}$ in $\mathcal{U}$: $\varphi \in \mathcal{T} \iff \neg\neg(\mathcal{T} \Vdash \varphi)$
- Pseudo Model Existence: if $\mathcal{T} \nvdash \varphi$ then there is $\mathcal{M}$ with $\neg\neg(\mathcal{M} \Vdash \mathcal{T})$ and $\mathcal{M} \nVdash \varphi$
- Quasi-Completeness: if $\mathcal{T} \Vdash \varphi$ then $\neg\neg(\mathcal{T} \vdash \varphi)$ (also since DNS $\iff \neg\neg$LEM)
- Completeness: needs MP/LEM depending on theory complexity and syntax fragment

Backwards Analysis

Two proofs of Quasi-Completeness from incomparable principles...

Fact

Model Existence implies WLEM.

Proof.

Given P, use model existence on $\mathcal{T} := \{x_0 \vee \neg x_0\} \cup \{x_0 \mid P\} \cup \{\neg x_0 \mid \neg P\}$. We have $\mathcal{T} \not\vdash \bot$ so if $\mathcal{M} \Vdash \mathcal{T}$, then either $\mathcal{M} \Vdash x_0$ or $\mathcal{M} \Vdash \neg x_0$, so either $\neg\neg P$ or $\neg P$, respectively. □

Fact

Quasi-Completeness implies the following principle: $\forall p : \mathbb{N} \to \mathbb{P}. \neg\neg(\forall n. \neg p\, n \vee \neg\neg p\, n)$

Proof.

Using similar tricks for $\mathcal{T} := \{x_n \vee \neg x_n\} \cup \{x_n \mid p\, n\} \cup \{\neg x_n \mid \neg p\, n\}$. □

Obvious consequence both from WLEM and DNS, maybe enough for Quasi-Completeness?

Countable Weak Excluded-Middle Shift[1]

$$\mathsf{WLEMS}_{\mathbb{N}} := \forall p : \mathbb{N} \to \mathbb{P}.\, (\forall n.\, \neg\neg(\neg p\, n \vee \neg\neg p\, n)) \to \neg\neg(\forall n.\, \neg p\, n \vee \neg\neg p\, n)$$

$$\Leftrightarrow \forall pq : \mathbb{N} \to \mathbb{P}.\, (\forall n.\, \neg\neg(\neg p\, n \vee \neg q\, n)) \to \neg\neg(\forall n.\, \neg p\, n \vee \neg q\, n)$$

Lemma

Assuming $\mathsf{WLEMS}_{\mathbb{N}}$, every stable quasi-prime theory is not not prime.

Proof.

Assume $\mathcal{T}$ not prime and derive a contradiction. Given the negative goal, from $\mathsf{WLEMS}_{\mathbb{N}}$ we obtain $\forall \varphi.\, \neg(\varphi \in \mathcal{T}) \vee \neg\neg(\varphi \in \mathcal{T})$. This yields exactly the instances of WLEM needed to derive that $\mathcal{T}$ is prime, contradiction. □

Already this lemma turns out to be enough for Quasi-Completeness!

[1] Mentioned in systematic study by Umezawa (1959) but absent from the literature otherwise

Quasi-Completeness via $\text{WLEMS}_{\mathbb{N}}$

Refined proof outline using $\text{WLEMS}_{\mathbb{N}}$:

- Lindenbaum Extension: if $\mathcal{T} \nvdash \varphi$ then there is stable not not prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash \varphi$
- Universal Model $\mathcal{U}$: consistent stable prime theories related by inclusion
- Truth Lemma for $\mathcal{T}$ in $\mathcal{U}$: $\varphi \in \mathcal{T} \iff \mathcal{T} \Vdash \varphi$
- Quasi Model Existence: if $\mathcal{T} \nvdash \varphi$ then there not not is $\mathcal{M}$ with $\mathcal{M} \Vdash \mathcal{T}$ and $\mathcal{M} \nVdash \varphi$
- Quasi-Completeness: if $\mathcal{T} \Vdash \varphi$ then $\neg\neg(\mathcal{T} \vdash \varphi)$
- Completeness: needs MP/LEM depending on theory complexity and syntax fragment

Consequences and Generalisation

Consequences:

- WLEM and Model Existence are equivalent
- $\mathsf{WLEMS}_{\mathbb{N}}$, Quasi Model Existence, and Quasi-Completeness are equivalent
- Completeness regarding enumerable $\mathcal{T}$ is equivalent to WLEMS + MP

Generalisation:

- Classical propositional logic
- Classical first-order logic, maybe intuitionistic first-order logic
- Classical and intuitionistic modal logics
- Bi-intuitionistic logic (depending on exclusion semantics)

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