

# Synthetic Incompleteness and Undecidability of Second-Order Logic

First Bachelor Seminar Talk

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Many first-order undecidability results formalized in the library of undecidability proofs [Forster et al., 2020]. For example:

- Validity
- Peano arithmetic
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All results in this talk are formalized in Coq.



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## Definition (Syntax)

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$$\varphi, \psi ::= \perp \mid P \mathbf{t} \mid \varphi \dot{\rightarrow} \psi \mid \varphi \dot{\wedge} \psi \mid \varphi \dot{\vee} \psi \mid \quad (P : \Sigma_P)$$
$$\dot{\vee} \varphi \mid \dot{\exists} \varphi$$
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Consider axiomatisation of Peano/Heyting arithmetic  
over signature  $(0, S, +, \cdot, \equiv)$ :

**Zero Addition** :  $\forall x. 0 + x \equiv x$

**Addition Recursion** :  $\forall xy. (Sx) + y \equiv S(x + y)$

**Disjointness** :  $\forall x. 0 \equiv Sx \rightarrow \perp$

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vs.

**PA<sub>1</sub>-Induction scheme** :  $\varphi[0] \rightarrow (\forall x. \varphi[x] \rightarrow \varphi[Sx]) \rightarrow \forall x. \varphi[x]$  (for all  $\varphi$ )

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Using the induction axiom, we can easily show that  $\cong$  is bijective and a homomorphism. Thus  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are isomorphic.

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- Under Unique Choice both semantics are equivalent.

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Possible solution:  $\mathcal{M} \models_\rho \dot{\exists}_f^n \varphi := \exists f^{D^{n+1} \rightarrow \mathbb{P}}. \text{total } f \wedge \text{functional } f \wedge \mathcal{M} \models_{f.\rho} \varphi$

- Under Unique Choice both semantics are equivalent.
- Luckily, our forthcoming reduction does not use function quantifiers, so it does not matter.

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## Deductive Incompleteness

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By Post's theorem [Bauer, 2006, Forster et al., 2019] it suffices

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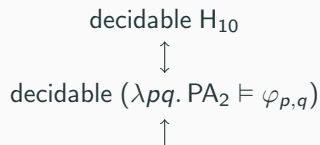
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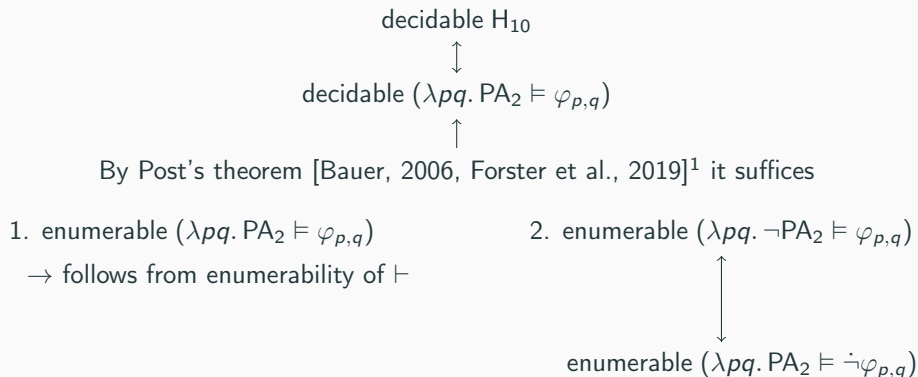
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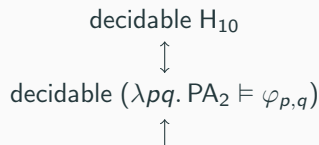


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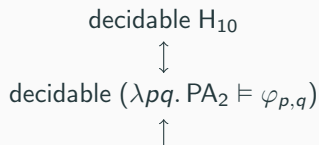
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- This deduction system would be complete for Henkin semantics.
- Further work on  $PA_2$  or  $ZF_2$  (incompleteness, conservativity, etc.)
- Connection between SOL and meta logic (e.g. inheritance of AC)



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$$\rho_f : \mathbb{N} \rightarrow \forall n. D^n \rightarrow D$$

$$(f \cdot \rho_f) 0 n := \begin{cases} f & \text{if } f \text{ has arity } n \\ \rho_f 0 n & \text{otherwise} \end{cases}$$

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# Undecidability of Validity

$$\begin{aligned} \forall \mathcal{M}. \mathcal{M} \models \forall f_0 f_S f_+ f_\times P_{\equiv}. PA'_2 &\dot{\rightarrow} \varphi'_{p,q} \\ \updownarrow & \\ \forall \mathcal{M}. \mathcal{M} \models PA_2 &\rightarrow \mathcal{M} \models \varphi_{p,q} \end{aligned}$$

## Lemma

---

$p = q$  has a solution iff  $\mathcal{M} \models \varphi_{p,q}$  for all models with  $\mathcal{M} \models PA_2$ .

## Proof.

---

$\rightarrow$ : Two possible proofs:

- If  $p = q$  has a solution, then  $\mathbb{N} \models \varphi_{p,q}$ . By categoricity it holds for all models of  $PA_2$ .
- Translate  $p = q$  solution to  $\mathcal{M}$  using a homomorphism  $\mu : \mathbb{N} \rightarrow \mathcal{M}$ .

$\leftarrow$ : Instantiate  $\mathcal{M}$  with standard model  $\mathbb{N}$  to obtain  $\mathbb{N} \models \varphi_{p,q}$ .

# Undecidability of Satisfiability

$$\begin{aligned} \exists \mathcal{M} \rho. \mathcal{M} \models_{\rho} \exists f_0 f_S f_+ f_{\times} P_{\equiv}. PA'_2 \wedge \varphi'_{p,q} \\ \updownarrow \\ \exists \mathcal{M} \rho. \mathcal{M} \models PA_2 \wedge \mathcal{M} \models_{\rho} \varphi_{p,q} \end{aligned}$$

## Lemma

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$p = q$  has a solution iff there is a model  $\mathcal{M} \models PA_2$  and  $\rho$  such that  $\mathcal{M} \models_{\rho} \varphi_{p,q}$ .

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---

$\rightarrow$ : If  $p = q$  has a solution, then the standard model  $\mathbb{N}$  fulfils  $\mathbb{N} \models \varphi_{p,q}$ .

$\leftarrow$ : If  $\mathcal{M} \models_{\rho} \varphi_{p,q}$  then also  $\mathbb{N} \models \varphi_{p,q}$  by categoricity.

Note that categoricity was required here, whereas it is optional for validity.

$$\frac{A[\uparrow_f^n] \vdash \varphi}{A \vdash \dot{\forall}_f^n \varphi} \text{Al}_f$$

$$\frac{A \vdash \dot{\forall}_f^n \varphi}{A \vdash \varphi[f]} \text{AE}_f$$

$$\frac{A \vdash \varphi[f]}{A \vdash \dot{\exists}_f^n \varphi} \text{El}_f$$

$$\frac{A \vdash \dot{\exists}_f^n \varphi \quad A[\uparrow_f^n], \varphi \vdash \psi[\uparrow_n]_f}{A \vdash \psi} \text{EE}_f$$

$$\frac{}{\dot{\exists}_p^n P. \dot{\forall} x_1 \dots x_n. P(x_1, \dots, x_2) \leftrightarrow \varphi[\uparrow_p^n]} \text{Compr}$$