Synthetic Incompleteness and Undecidability of Second-Order Logic

First Bachelor Seminar Talk

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Introduction

Many first-order undecidability results formalized in the library of undecidability proofs [Forster et al., 2020]. For example:

- Validity
- Peano arithmetic
- Satisfiability
- Zermelo–Fraenkel set theory

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All results in this talk are formalized in Coq.

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Definition (Syntax)

$$\varphi, \psi ::= \dot{\bot} \mid P\ \boldsymbol{t} \mid \varphi \dot{\to} \psi \mid \varphi \dot{\land} \psi \mid \varphi \dot{\lor} \psi \mid \dot{\forall}\, \varphi \mid \dot{\exists}\, \varphi \qquad (P : \Sigma_P)$$

$$t ::= \mathsf{var}_x \mid f\ \boldsymbol{t} \qquad (x : \mathbb{V},\ f : \Sigma_f)$$

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Peano Arithmetic

Consider axiomatisation of Peano/Heyting arithmetic over signature $(0, S, +, \cdot, \equiv)$:

Zero Addition : $\dot\forall x.\, 0 + x \equiv x$

Addition Recursion : $\dot\forall xy.\, (Sx) + y \equiv S(x + y)$

Disjointness : $\dot\forall x.\, 0 \equiv Sx \dot\to \dot\bot$

Equlity Reflexive : $\dot\forall x.\, x \equiv x$

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$$\textbf{PA}_2\textbf{-Induction} : \dot{\forall}^1_p P.\, P\,0 \dot{\to} (\dot{\forall} x.\, P\,x \dot{\to} P\,(Sx)) \dot{\to} \dot{\forall} x.\, P\,x$$

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PA$_1$-Induction scheme : $\varphi[0] \mathrel{\dot\to} (\dot\forall x.\, \varphi[x] \mathrel{\dot\to} \varphi[Sx]) \mathrel{\dot\to} \dot\forall x.\, \varphi[x]$ (for all φ)

Semantics

Definition (Model)

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Using the induction axiom, we can easily show that $\cong$ is bijective and a homomorphism. Thus $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic.

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Compuational Semantics

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- Under Unique Choice both semantics are equivalent.

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Possible solution: $\mathcal{M} \vDash_\rho \dot{\exists}^n_f\, \varphi \;:=\; \exists f^{D^{n+1} \to \mathbb{P}}.\, \mathsf{total}\, f \wedge \mathsf{functional}\, f \wedge \mathcal{M} \vDash_{f \cdot \rho} \varphi$

- Under Unique Choice both semantics are equivalent.
- Luckily, our forthcoming reduction does not use function quantifiers, so it does not matter.

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Lemma

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- $p = q$ has a solution iff $\exists \mathcal{M} \rho.\, \mathcal{M} \vDash_\rho \dot{\exists}\, f_0\, f_S\, f_+\, f_\times\, P_\equiv.\, \mathsf{PA}'_2 \mathrel{\dot{\wedge}} \varphi'_{p,q}$.

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- Further work on PA_2 or ZF_2 (incompleteness, conservativity, etc.)
- Connection between SOL and meta logic (e.g. inheritance of AC)

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Environments

$$\rho_f : \mathbb{N} \to \forall n.\, D^n \to D$$

$$(f \cdot \rho_f)\ 0\ n := \begin{cases} f & \text{if } f \text{ has arity } n \\ \rho_f\ 0\ n & \text{otherwise} \end{cases}$$

$$(f \cdot \rho_f)\ (Sx)\ n := \begin{cases} \rho_f\ x\ n & \text{if } f \text{ has arity } n \\ \rho_f\ (Sx)\ n & \text{otherwise} \end{cases}$$

Undecidability of Validity

$$\forall \mathcal{M}.\, \mathcal{M} \vDash \dot{\forall}\, f_0\, f_S\, f_+\, f_\times\, P_\equiv.\, \mathsf{PA}_2' \mathrel{\dot{\rightarrow}} \varphi'_{p,q}$$
$$\updownarrow$$
$$\forall \mathcal{M}.\, \mathcal{M} \vDash \mathsf{PA}_2 \rightarrow \mathcal{M} \vDash \varphi_{p,q}$$

Lemma

$p = q$ has a solution iff $\mathcal{M} \vDash \varphi_{p,q}$ for all models with $\mathcal{M} \vDash \mathsf{PA}_2$.

Proof.

$\rightarrow$: Two possible proofs:

- If $p = q$ has a solution, then $\mathbb{N} \vDash \varphi_{p,q}$. By categoricity it holds for all models of PA_2.
- Translate $p = q$ solution to $\mathcal{M}$ using a homomorphism $\mu : \mathbb{N} \to \mathcal{M}$.

$\leftarrow$: Instantiate $\mathcal{M}$ with standard model $\mathbb{N}$ to obtain $\mathbb{N} \vDash \varphi_{p,q}$.

Undecidability of Satisfiability

$$\exists \mathcal{M} \rho.\, \mathcal{M} \vDash_\rho \dot{\exists}\, f_0\, f_S\, f_+\, f_\times\, P_\equiv.\, \mathsf{PA}_2' \,\dot{\wedge}\, \varphi'_{p,q}$$
$$\updownarrow$$
$$\exists \mathcal{M} \rho.\, \mathcal{M} \vDash \mathsf{PA}_2 \wedge \mathcal{M} \vDash_\rho \varphi_{p,q}$$

Lemma

$p = q$ has a solution iff there is a model $\mathcal{M} \vDash \mathsf{PA}_2$ and ρ such that $\mathcal{M} \vDash_\rho \varphi_{p,q}$.

Proof.

$\rightarrow$: If $p = q$ has a solution, then the standard model $\mathbb{N}$ fulfils $\mathbb{N} \vDash \varphi_{p,q}$.

$\leftarrow$: If $\mathcal{M} \vDash_\rho \varphi_{p,q}$ then also $\mathbb{N} \vDash \varphi_{p,q}$ by categoricity.

Note that categoricity was required here, whereas it is optional for validity.

Natural Deduction

$$\frac{A[\uparrow^n_f] \vdash \varphi}{A \vdash \dot{\forall}^n_f \, \varphi} \; \mathsf{AI_f} \qquad \frac{A \vdash \dot{\forall}^n_f \, \varphi}{A \vdash \varphi[f]} \; \mathsf{AE_f}$$

$$\frac{A \vdash \varphi[f]}{A \vdash \dot{\exists}^n_f \, \varphi} \; \mathsf{EI_f} \qquad \frac{A \vdash \dot{\exists}^n_f \, \varphi \qquad A[\uparrow^n_f], \varphi \vdash \psi[\uparrow_n]_f}{A \vdash \psi} \; \mathsf{EE_f}$$

$$\frac{}{\dot{\exists}^n_p \, P. \, \dot{\forall} x_1 \ldots x_n. \, P(x_1, \ldots, x_2) \mathrel{\dot{\leftrightarrow}} \varphi[\uparrow^n_p]} \; \mathsf{Compr}$$