Mechanizing Second-Order Logic in Coq

Final Bachelor Talk

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First-order logic

Quantification over individuals:

$$\forall x.\, \exists y.\, y > x$$

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- Mechanize undecidability results for SOL

Related work

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- Synthetic undecidability and incompleteness of axiom systems by [Kirst and Hermes, 2021]
- Synthetic computability theory [Bauer, 2006, Forster et al., 2019a] as employed in the Coq Library of Undecidability Proofs [Forster et al., 2019b].

Definition (Syntax)

$$t ::= x_i \mid \mathcal{F}\ \boldsymbol{t} \qquad (\mathcal{F} : \Sigma_f)$$

$$\varphi, \psi ::= \dot{\bot} \mid \mathcal{P}\ \boldsymbol{t} \mid p_i^n\ \boldsymbol{t} \mid \varphi \mathrel{\dot{\to}} \psi \mid \varphi\ \dot{\wedge}\ \psi \mid \varphi\ \dot{\vee}\ \psi \qquad (\mathcal{P} : \Sigma_p)$$

$$\mid \dot{\forall}\ \varphi \mid \dot{\exists}\ \varphi \mid \dot{\forall}_p^n\ \varphi \mid \dot{\exists}_p^n\ \varphi \qquad (i, n : \mathbb{N})$$

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$$\rho \vDash \dot{\forall}_p^n \varphi \ :=\ \forall P^{D^n \to \mathsf{Prop}}.\ P \cdot \rho \vDash \varphi$$

Second-order Peano Arithmetic

Standard semantics can uniquely characterize the numbers via PA_2:

Zero Addition : $\dot{\forall} x.\, 0 + x \equiv x$

Addition Recursion : $\dot{\forall} xy.\, (Sx) + y \equiv S(x + y)$

Disjointness : $\dot{\forall} x.\, 0 \equiv Sx \mathrel{\dot{\to}} \dot{\bot}$

Equlity Reflexive : $\dot{\forall} x.\, x \equiv x$

Zero Multiplication : $\dot{\forall} x.\, 0 \cdot x \equiv 0$

Multiplication Recursion : $\dot{\forall} xy.\, (Sx) \cdot y \equiv y + x \cdot y$

Successor Injective : $\dot{\forall} xy.\, Sx \equiv Sy \mathrel{\dot{\to}} x \equiv y$

Equlity Symmetric : $\dot{\forall} xy.\, x \equiv y \mathrel{\dot{\to}} y \equiv x$

Induction : $\dot{\forall} P.\, P(0) \mathrel{\dot{\to}} (\dot{\forall} x.\, P(x) \mathrel{\dot{\to}} P(Sx)) \mathrel{\dot{\to}} \dot{\forall} x.\, P(x)$

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Theorem (Categoricity)

All models of PA_2 are isomorphic. We say that PA_2 is **categorical**.

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Failure of downwards direction via categoricity of second-order real analysis [Shapiro, 1991] or set theory [Kirst and Smolka, 2017]

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- $\mathcal{T}_{\neq}$ itself does not have a model. Otherwise, $\mathbb{N}$ would also need to be a model because of categoricity which is not possible. □

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  However requires much more involved proof!

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Enumerator for $\lambda\varphi.\, \mathbb{N} \vDash \varphi$ would yield decider for H_{10} via Post's theorem:

- Enumerate H_{10} via $\mathbb{N} \vDash \varphi_{p,q}$.
- Enumerate $\overline{\mathrm{H}_{10}}$ via $\mathbb{N} \vDash \dot{\neg}\varphi_{p,q}$.

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Theorem (Finitary Incompleteness)

Existence of a sound, enumerable and finitary complete deduction system $\vdash : \mathcal{L}(\text{form}) \to \text{form} \to \text{Prop}$ implies decidability of the halting problem under MP.

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Proof.

$\vdash$ yields enumerator for $\lambda\varphi.\text{PA}_2 \vDash \varphi$ and thus all truths in $\mathbb{N}$. $\square$

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Corollary (Non-Enumerability).

Those problems are also not enumerable under MP.

Henkin Semantics

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Definition (Henkin Semantics).

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$\mathbb{U}_n$ should satisfy comprehension, i.e. it must at least contain all second-order definable properties.

Reduction to FOL [Nour and Raffalli, 2003]

Turn φ into $\varphi^\star$ by replacing predicate quantifiers with individual ones:

$$\dot{\forall} x.\, \dot{\exists} P.\, P(x, x) \quad \rightsquigarrow \quad \dot{\forall} x.\, \dot{\exists} p.\, \mathrm{App}_2(p, x, x)$$

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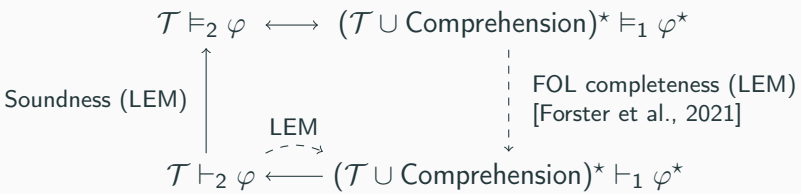

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We combine upward and downward in one property: "*Every theory with an infinite model has models of every infinite cardinality.*"

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$\Rightarrow$ No categorical axiomatization of $\mathbb{N}$ possible!

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Contributions: To the best of our knowledge, first mechanization of SOL.

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Semantics	$\mathbb{N}$ Categorical	Completeness	Compactness	Löwenheim-Skolem
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- Formalization of categoricity worked relatively smoothly
- Henkin reduction by far the most difficult part ($\sim$ 2,000 LOC)
  $\Rightarrow$ Especially handling of de Bruijn encoding challenging

Future Work

- Merge development into Coq Library of Undecidability Proofs [Forster et al., 2019b]

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Undecidability of Validity

$$\forall \mathcal{M}.\, \mathcal{M} \vDash \dot{\forall}\, f_0\, f_S\, f_+\, f_\times\, P_\equiv.\, \mathrm{PA}_2' \mathrel{\dot{\to}} \varphi'_{p,q}$$
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$$\forall \mathcal{M}.\, \mathcal{M} \vDash \mathrm{PA}_2 \to \mathcal{M} \vDash \varphi_{p,q}$$

Lemma

$p = q$ has a solution iff $\mathcal{M} \vDash \varphi_{p,q}$ for all models with $\mathcal{M} \vDash \mathrm{PA}_2$.

Proof.

$\to$: Two possible proofs:

- If $p = q$ has a solution, then $\mathbb{N} \vDash \varphi_{p,q}$. By categoricity it holds for all models of PA_2.
- Translate $p = q$ solution to $\mathcal{M}$ using a homomorphism $\mu : \mathbb{N} \to \mathcal{M}$.

$\leftarrow$: Instantiate $\mathcal{M}$ with standard model $\mathbb{N}$ to obtain $\mathbb{N} \vDash \varphi_{p,q}$.

Undecidability of Satisfiability

$$\exists \mathcal{M} \rho.\, \mathcal{M} \vDash_\rho \dot{\exists}\, f_0\, f_S\, f_+\, f_\times\, P_\equiv.\, \mathsf{PA}_2' \mathbin{\dot{\wedge}} \varphi'_{p,q}$$
$$\updownarrow$$
$$\exists \mathcal{M} \rho.\, \mathcal{M}, \rho \vDash \mathsf{PA}_2 \wedge \mathcal{M}, \rho \vDash \varphi_{p,q}$$

Lemma

$p = q$ has a solution iff there is a model $\mathcal{M} \vDash \mathsf{PA}_2$ and ρ such that $\mathcal{M} \vDash_\rho \varphi_{p,q}$.

Proof.

$\rightarrow$: If $p = q$ has a solution, then the standard model $\mathbb{N}$ fulfils $\mathbb{N} \vDash \varphi_{p,q}$.

$\leftarrow$: If $\mathcal{M}, \rho \vDash \varphi_{p,q}$ then also $\mathbb{N} \vDash \varphi_{p,q}$ by categoricity.

Note that categoricity was required here, whereas it is optional for validity.

Non-Enumerability

Theorem.

Enumerability of validity in PA_2 implies decidability of the halting problem under MP.

Proof.

- Enumerate H_{10} via $\mathrm{PA}_2 \vDash \varphi_{p,q}$.
- Enumerate $\overline{\mathrm{H}_{10}}$ via $\neg\mathrm{PA}_2 \vDash \varphi_{p,q} \;\leftrightarrow\; \mathrm{PA}_2 \vDash \dot{\neg}\varphi_{p,q}$ (Categoricity).

Yields decider via Post's theorem. □

Natural Deduction

$$\frac{A[\uparrow_p^n] \vdash_2 \varphi}{A \vdash_2 \dot{\forall}_p^n \varphi}\ \mathrm{AI_p} \qquad \frac{A \vdash_2 \dot{\forall}_p^n \varphi}{A \vdash_2 \varphi[P]}\ \mathrm{AE_p}$$

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$$\frac{}{A \vdash_2 \dot{\exists}_p^n P.\, \dot{\forall} x_1 \ldots x_n.\, P(x_1, \ldots, x_2) \dot{\leftrightarrow} \varphi[\uparrow_p^n]}\ \mathrm{Compr}_p$$

Semantic Henkin Reduction

- Turn Henkin model $\mathcal{H}$ into first-order model $\mathcal{H}^\star$ with $D^\star := D \cup \mathbb{U}$ and $\mathsf{App}_n\,(x :: \boldsymbol{v}) := \mathsf{toPred}_n\,x\,(\mathsf{toIndi}\,\boldsymbol{v})$

$$\mathcal{H} \vDash_2 \varphi \;\leftrightarrow\; \mathcal{H}^\star \vDash_1 \varphi^\star$$

- Turn first-order model $\mathcal{M}$ into Henkin model $\mathcal{M}^\diamond$ with $D^\diamond := D$ and $\mathbb{U}$ induces by interpretation of App.

$$\mathcal{M} \vDash_1 \mathsf{Comprehension}^\star \;\to\; \mathcal{M}^\diamond \vDash_2 \varphi \;\leftrightarrow\; \mathcal{M} \vDash_1 \varphi^\star$$

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Backwards Translation

Define a backwards translation $\_^{\diamond} : \mathsf{form}_1(\Sigma_+) \to \mathsf{form}_2(\Sigma_{\mathsf{err}})$. For example

$$(\forall x.\, \mathsf{predApp}_0(x) \mathbin{\dot\wedge} \mathsf{predApp}_1(x, x))^{\diamond}$$

$$||$$

$$\forall x_i.\, \forall^0_p\, x^0_p.\, \forall^1_p\, x^1_p.\, x^0_p \mathbin{\dot\wedge} x^1_p(x_i)$$

$$(\mathsf{predApp}_1(f(x), y))^{\diamond} = \mathsf{Err}_1(y_i)$$

Special error symbol if first argument is not a variable

Completeness

Lemma

1. $A \vdash_1 \varphi \;\rightarrow\; A^{\diamond} \vdash_2^{\mathsf{err}} \varphi^{\diamond}$
2. $\vdash_2 \varphi^{\star\diamond} \dot{\leftrightarrow} \varphi$

$$\mathcal{T} \vDash_2 \varphi \;\longleftrightarrow\; \mathcal{T}^{\star}, \mathcal{C} \vDash_1 \varphi^{\star}$$

FOL Completeness [Forster et al., 2021]

LEM

$$\mathcal{T}^{\star}, \mathcal{C} \vdash_1 \varphi^{\star} \xrightarrow{(1)} \mathcal{T}^{\star\diamond}, \mathcal{C}^{\diamond} \vdash_2^{\mathsf{err}} \varphi^{\star\diamond}$$

Error symbol can occur in this derivation!

Remove error symbol using comprehension

$$\mathcal{T}^{\star\diamond}, \mathcal{C}^{\diamond} \vdash_2 \varphi^{\star\diamond} \xleftrightarrow{(2)} \mathcal{T}, \mathcal{C}^{\diamond} \vdash_2 \varphi \xrightarrow[\vdash_2 \text{ proves comprehension}]{} \mathcal{T} \vdash_2 \varphi$$

Internal Categoricity [Väänänen and Wang, 2012]

Consider a theory $\mathcal{T}$ depending on a single predicate symbol $\mathcal{P}$

$$\mathsf{Categ}(\mathcal{T}) := \dot{\forall} D_1 D_2 P_1 P_2 . \mathcal{T}(P_1)^{D_1} \dot{\to} \mathcal{T}(P_2)^{D_2} \dot{\to} \dot{\exists} \cong . \mathsf{Iso}(\cong, D_1, D_2, P_1, P_2)$$

where $\mathcal{T}(P_1)^{D_1}$ replaces $\mathcal{P}$ with the variable P_1 and guards all quantifiers with the domain predicate D_1.

- $\mathcal{T}$ is categorical iff $\vDash \mathsf{Categ}(\mathcal{T})$
- Provable in many cases (despite incompleteness), e.g. $\vdash \mathsf{Categ}(\mathsf{PA}_2)$.
  - $\Rightarrow$ Categoricity can be written and proven at the object level, without depending on any external set theory (or type theory in our case)