A Coq Library for Finite Types

2^{nd} bachelor seminar talk

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17. Juni 2016

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Finite types

What is a finite type?

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- Type

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- Completeness proof for list

FINITE TYPES

What is a finite type?

- Type
- Finite number of inhabitants

What do we have formally?

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- List of inhabitants
- Completeness proof for list
- Decidability of equality

in Coq

```
Class finTypeC (type:eqType): Type := FinTypeC {
enum: list type;
enum_ok: ∀ x: type, count enum x = 1
}.
```

IN COQ

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Class finTypeC (type:eqType): Type := FinTypeC {
enum: list type;
enum_ok: ∀ x: type, count enum x = 1
}.

 Structure finType: Type := FinType {
type :> eqType;
class: FinTypeC type
}.
```

Equivalence Principles

Finite Types satisfy important equivalences:

About: elem

elem is a projection from a finType to its list of elements

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$$(\forall\ (x : F), p\ x) \leftrightarrow \forall\ x \in (elem\ F), p\ x$$

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$$(\exists\ (x : F), p\ x) \leftrightarrow \exists\ x, x \in (elem\ F) \rightarrow p\ x$$

About: elem

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Decidability

Finiteness $\to$ decidability of quantifiers.

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Finiteness → decidability of quantifiers.

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Instance finType_forall_dec (F:finType) (p:F → ℙ):
(∀ x, dec(p x)) → dec (∀ x, p x).
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Proof: equivalence properties + corresponding instances for lists

de Morgan

de Morgan's laws:

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- $(\forall x : p\ x) \leftrightarrow \neg\ \exists x\ (\neg\ p\ x)$

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- $\neg\,(P \to Q) \to P \wedge \neg\, Q$
- $(\forall x : p\ x) \leftrightarrow \neg\, \exists x\ (\neg\, p\ x)$
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- $\neg(P \land Q) \leftrightarrow \neg P \lor \neg Q$
  - true if P or Q decidable
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Problem: instantiation of existential quantifiers

DE MORGAN

$(\exists x : p\ x) \leftrightarrow \neg \forall x\ (\neg\ p\ x)$

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Problem: instantiation of existential quantifiers

finTypes allow to decide $\exists x : p\ x$ and $\exists x : \neg\ p\ x$

CONSTRUCTIVE CHOICE

Normally: Elim Restriction for $\exists$

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But: One can construct function $\exists x,\ p\ x \rightarrow \{x \mid p\ x\}$

Try every element of the type.

One needs to fulfil p

Cardinality

Cardinality X: Number of inhabitants of X

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Definition Cardinality (F: finType) := |elem F|
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Cardinality Lemmas:

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Cardinality\ (F_1 \times F_2) &= Cardinality\ F_1 * Cardinality\ F_2 \\
Cardinality\ (option\ F) &= S(Cardinality\ F) \\
Cardinality\ (F_1 + F_2) &= Cardinality\ F_1 + Cardinality\ F_2 \\
Cardinality\ (F_1 \longrightarrow F_2) &= (Cardinality\ F_2)^{(Cardinality\ F_1)}
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\end{aligned}
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$$\mathit{Cardinality}\ F = \mathit{card}\ (\mathit{elem}\ F)$$

PIDGEONHOLE PRICIPLES

The following holds on finite Types:

$(\exists \text{ injection } f : F_1 \to F_2) \to \mathit{Cardinality}\ F_1 \leq \mathit{Cardinality}\ F_2$

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$(\exists \text{ bijection } f : F_1 \to F_2) \to \mathit{Cardinality}\ F_1 = \mathit{Cardinality}\ F_2$

PIDGEONHOLE PRINCIPLES: IMPORTANT LEMMAS

Reminder: card

```
Fixpoint card A: ℕ :=
match A with | nil ⇒ 0
| x::A′⇒ if decision x el A then card A else 1 + card A
end.
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$card\ A \leq |A|$

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$dupfree\ (elem\ F)$

$surjective\ f \to (elem\ F_2) \subseteq image\ f$

$A \subseteq B \to card\ A \leq card\ B$

---

surjection $(f : F_1 \to F_2)$

$$card\ |image\ f| \geq card\ |elem\ F_2|\ \checkmark$$

FIXED POINT ALGORITHMS

Definition (fixed point)

An argument x is a *fixed point* of a function *f* if $f\ x = x$.

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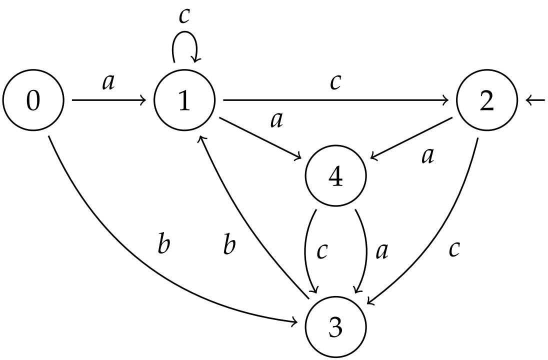
Special case: Compute subset of some set

Example: reachable states in NFAs



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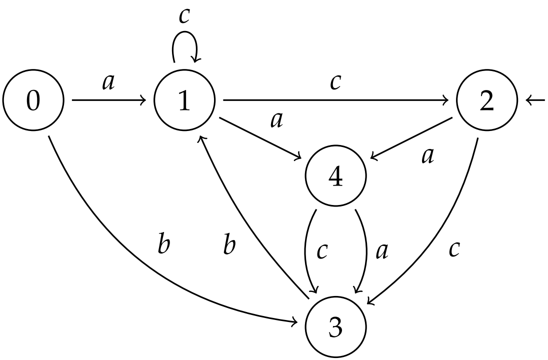
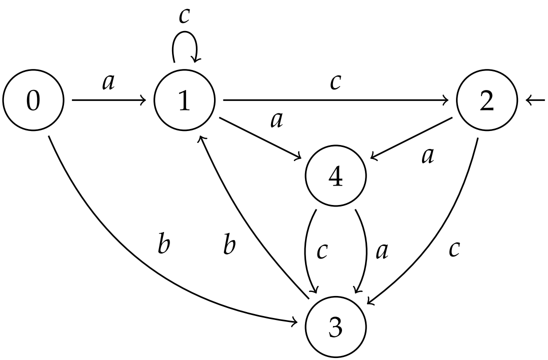


1. {}

Example: reachable states in NFAs



1. {}
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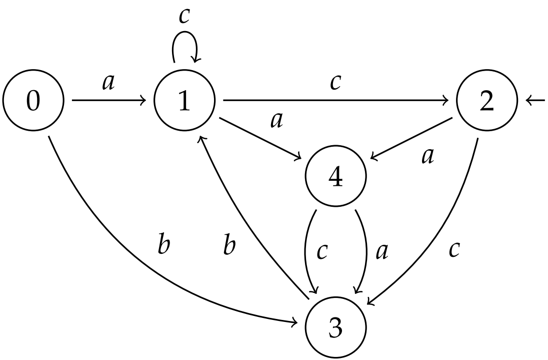
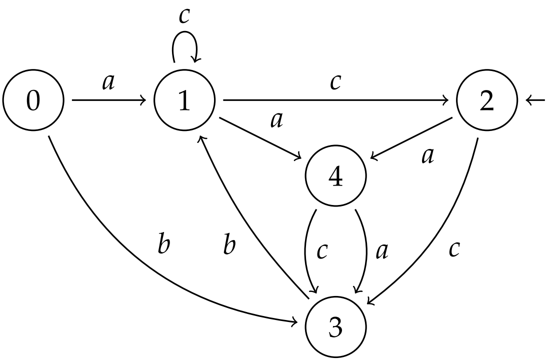
Example: reachable states in NFAs



1. {}
2. {2}
3. {2,3}

Example: reachable states in NFAs



1. {}
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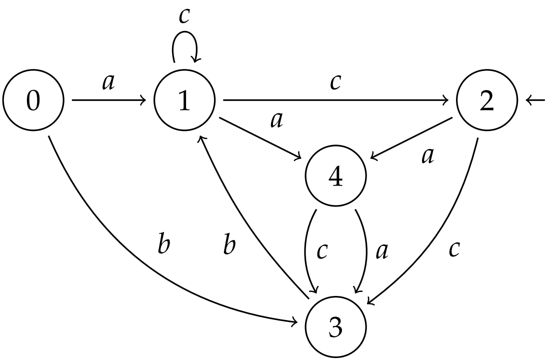
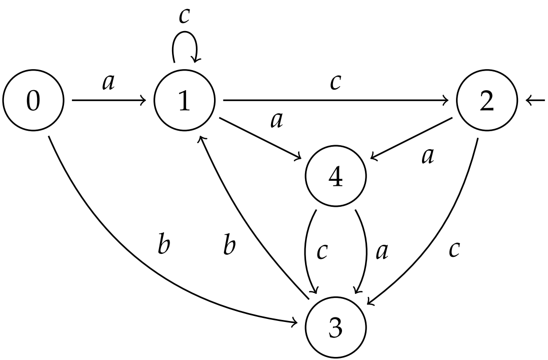
4. {2,3,4}

Example: reachable states in NFAs



1. {}
2. {2}
3. {2,3}
4. {2,3,4}
5. {2,3,4,1}

Example: reachable states in NFAs



1. {}
2. {2}
3. {2,3}
4. {2,3,4}
5. {2,3,4,1}
6. {2,3,4,1} ← fixed point

Finite closure iteration

Compute subset of finite type F

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- step: list $\mathrm{F} \to F \to \mathbb{P}$

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- function pick A : {x | step A x $\land \neg (x \in A)$} + $\forall x$, step A x $\to\ x \in A$.

Finite closure iteration

Compute subset of finite type F

- step: list F $\rightarrow F \rightarrow \mathbb{P}$
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  - decide ($\forall x, step\ A\ x \rightarrow x \in A$.)

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FCStep

```
Definition FCStep A :=
match (pick A) with
|inl L ⇒ match L with
         |exists _ x _ ⇒ x::A end
|inr _ ⇒ A end.
```

Finite closure iteration

Claim: for all lists A

```
iter (Cardinality F) FCStep A
```

is a fixed point of FCStep.

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Definition (monotone)

A function (f: list X → list X) is called *monotone* if a given list A is either a fixed-point of f or $card\ (f\ A) > card\ A$

Monotonicity

Claim: for all monotone functions f and lists A

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Proof Sketch.

Let f: *list* $F \to$ *list* F be monotone. Then by induction on $n \in \mathbb{N}$

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Proof Sketch.

Let f: *list* $F \rightarrow$ *list* F be monotone. Then by induction on $n \in \mathbb{N}$ every A: *list* F is either a fixed point of f

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iter (Cardinality F) f A
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is a fixed point of f.

Proof Sketch.

Let *f: list F* $\rightarrow$ *list F* be monotone. Then by induction on $n \in \mathbb{N}$ every *A: list F* is either a fixed point of *f* or *card (iter n f A)* $\geq n$.

Monotonicity

Claim: for all monotone functions f and lists A

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iter (Cardinality F) f A
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Let f: *list F* $\rightarrow$ *list F* be monotone. Then by induction on $n \in \mathbb{N}$ every A: *list F* is either a fixed point of f or *card (iter n f A)* $\geq n$.

Now choose n = *Cardinality F*.

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Now choose n = *Cardinality F*.
Assume *iter n f a* is no fixed point of f.

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or *card (iter* n f A*)* $\geq n$.

Now choose n = *Cardinality* F.
Assume *iter* n f a is no fixed point of f.
Then *card*(f (*iter* n f a)) > *Cardinality* F because f is monotone.

Monotonicity

Claim: for all monotone functions f and lists A

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iter (Cardinality F) f A
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is a fixed point of f.

Proof Sketch.

Let *f: list F* $\rightarrow$ *list F* be monotone. Then by induction on $n \in \mathbb{N}$ every *A: list F* is either a fixed point of *f* or *card (iter n f A)* $\geq n$.

Now choose n = *Cardinality F*.
Assume *iter n f a* is no fixed point of f.
Then *card(f (iter n f a))* > *Cardinality F* because f is monotone.
But *Cardinality F* = *card (elem F)* and *elem F* contains all possible elements of type F. ∎

FINITE CLOSURE ITERATION: AGAIN

Claim: for all lists A

```
iter (Cardinality F) FCStep A
```

is a fixed point of FCStep.

Proof Sketch.

By case analysis on *pick A*: FCStep is monotone. ■

Finite Closure iteration: Again

Claim: for all lists A

`iter (Cardinality F) FCStep A`

is a fixed point of FCStep.

Proof Sketch.

By case analysis on *pick A*: FCStep is monotone. ■

⇒ `iter (Cardinality F) FCStep A`
is the fixed point we want.

A glimpse into the crystal ball

What will the future bring?

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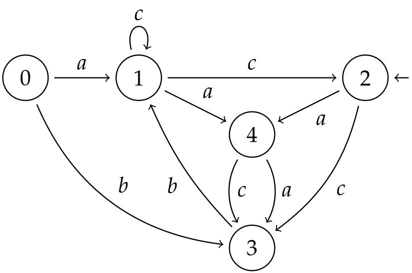

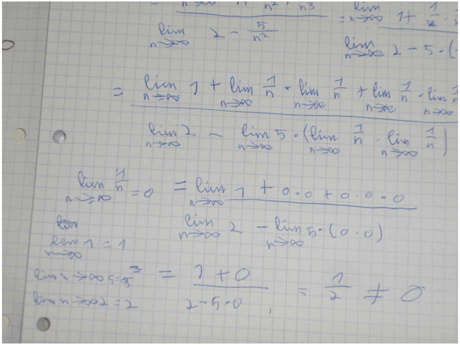

Sources and Inspiration

- Smolka, Gert and Brown, Chad E.
  Introduction to Computational Logic
  Lecture Notes SS 2014
- Smolka, Gert
  Base library for ICL
  Version: February 15^{th} 2016

The End

Thank you for your attention

Any questions? Ask away!