A Coq Library for Finite Types

Bachelor Talk

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COMPUTER SCIENCE

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Finite types

What is a finite type?

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- Type

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- Type
- Finite number of inhabitants

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Representation:

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- $|elem\ X|$

Three usability features

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  - Canonical structures
- We can compute Y out of its base type.
  - $\mathit{Cardinality}(\mathit{tofinType}\ \mathbb{B}) = 2$
  - Mainly type classes

Conversion to lists

$$(\forall\ (x : X), p\ x) \leftrightarrow \forall\ x \in (elem\ X), p\ x$$

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$$(\exists\ (x : X), p\ x) \leftrightarrow \exists\ x, x \in (elem\ X) \rightarrow p\ x$$

Classical properties

Finite types often behave classically:

From list conversions:

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- $(\exists\, x : X,\ p\ x) \leftrightarrow \neg\, \forall\, x : X,\ (\neg\, p\ x)$.
- $\neg\, (\forall\, x : X, p\ x) \leftrightarrow \exists\, x : X, \neg\, p\ x$.

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- $(\exists\, x : X,\ p\ x) \leftrightarrow \neg\, \forall\, x : X,\ (\neg\, p\ x)$.
- $\neg\, (\forall\, x : X, p\ x) \leftrightarrow \exists\, x : X, \neg\, p\ x$.
- There is a constructive choice function $\exists\, x : X, p\ x \to \{x : X \mid p\ x\}$.

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- $\mathit{Cardinality}\ X = \mathit{card}\ (\mathit{elem}\ X)$.
- $\mathit{Cardinality}\ X \geq \mathit{card}\ A$.
- $\mathit{Cardinality}\ X \geq |A|$, if A is duplicate free.

Pigeonhole principles

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$(\exists \text{ bijection } f : X_1 \to X_2) \to \mathit{Cardinality}\ X_1 = \mathit{Cardinality}\ X_2$

Vectors

Usually: Collection of objects of some "type" with fixed size.

$$\mathbb{R}^n : \begin{pmatrix} \pi \\ e \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix}$$

indexed by some number n

Vectors

Now: Collection of objects of some type Y with fixed size.

$$elem\ X := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{X-indexed Y vector} : \begin{pmatrix} y_1 & 1 \\ y_2 & 2 \\ \vdots & \vdots \\ y_n & n \end{pmatrix}$$

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In Coq dependent pair: $\{ A \mid |A| = Cardinality\ X \}$

Vectors

Function interpretation:

$$elem\ X := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{matrix} \longrightarrow \\ \longrightarrow \\ \vdots \\ \longrightarrow \end{matrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} =: f$$

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$$X \longrightarrow Y := X\text{-indexed } Y \text{ vector.}$$

$$image\ f := [y_1, y_2, \ldots, y_n]$$

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We want to decide equality

In Coq dependent pair: $\{A \mid |A| = \mathit{Cardinality}\ X\}$

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A predicate $p : X \to \mathbb{P}$ is called *pure* if for every $x{:}X$ there is only one proof of $p\ x$.

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$\Rightarrow$ Vectors are discrete and extensional:

Fact

Let f and g be two vectors. Then $\mathit{image}\ f = \mathit{image}\ g \to f = g$.

A finite vector type

Theorem

There is a finite type $X_2^{X_1}$ such that $X_2^{X_1} = X_1 \longrightarrow X_2$.

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Theorem

$\mathit{Cardinality}\ X_2^{X_1} = (\mathit{Cardinality}\ X_2)^{\mathit{Cardinality}\ X_1}$.

Vectors vs. functions

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f vector $X \longrightarrow Y$

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Function *applyVect:* $X \longrightarrow Y \to X \to Y$.

VECTORS VS. FUNCTIONS

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Function *applyVect:* $X \longrightarrow Y \to X \to Y$.

Defined as coercion. We can write $f\ x$ instead of $applyVect\ f\ x$.

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Function *vectorise:* $(X \to Y) \to (X \longrightarrow Y)$.

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applyVect and *vectorise* are inverse functions:

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Let f be a vector $X \longrightarrow Y$. Then *vectorise* $f = f$.

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applyVect and *vectorise* are inverse functions:

Theorem

Let f be a vector $X \longrightarrow Y$. Then $\mathit{vectorise}\ f = f$.

Theorem

Let f be a function $X \to Y$. Then $(\mathit{vectorise}\ f)\ x = f\ x$.

More compount types

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FINITE CLOSURE ITERATION[14, 13]

Compute subset of finite type F:

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FCStep

```
Definition FCStep A :=
match (pick A) with
|inl L ⇒ match L with
          |exists _ x _ ⇒ x::A end
|inr _ ⇒ A end.
```

Finite closure iteration[14, 13]: Idea

Iterate *FCStep* until it reaches a fixed point

Finite closure iteration[14, 13]: Idea

Iterate *FCStep* until it reaches a fixed point

How many times?

Fact

Let A be a list over X and `FCIter := FCStep`$^{\texttt{Cardinality X}}$. Then `FCIter A` is a fixed point of *FCStep*.

FCIter induction [14, 13]

Induction principle for predicates preserved by *FCStep*:

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Induction principle for predicates preserved by *FCStep*:
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Theorem

Let p be a predicate over X and A a list over X. Then
$A \subseteq p \to (\forall\, A\ x, A \subseteq p \to \mathit{step}\ A\ x \to p\ x) \to \mathit{FCIter}\ A \subseteq p.$

Least fixed points

Corollary

Let A be a list over X. Then `FCIter A` is a fixed point of *FCStep*.

Is it a least fixed point?

Least fixed points

Corollary

Let A be a list over X. Then `FCIter A` is a fixed point of *FCStep*.

Is it a least fixed point?

No! But ...

Least fixed points containing A

Definition (Least fixed points containing A)

Let A: *list* Y and f: *list* $Y \to$ *list* Y. A fixed point B of f is called the *least fixed point containing* A if $A \subseteq B$ and for any other fixed point B' of f: $A \subseteq B' \to B \subseteq B'$.

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A *step* predicate is called *consistent* if
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Theorem

If *step* is consistent then for any A the list *FCIter* A is the least fixed point containing A.

Test case: Finite automata with finite types

Inspired by [3] (Talk on Monday 3:15 pm)

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- Some s of type S, the start state.

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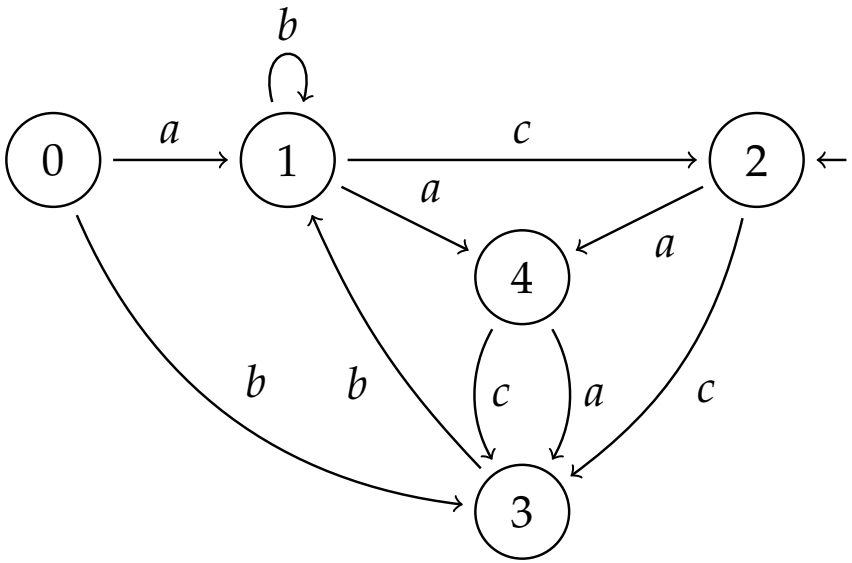
Definition (Acceptance)

An automaton *accepts* a word w if $F\ (\delta_S^*\ s\ w)$.

FCIter in action



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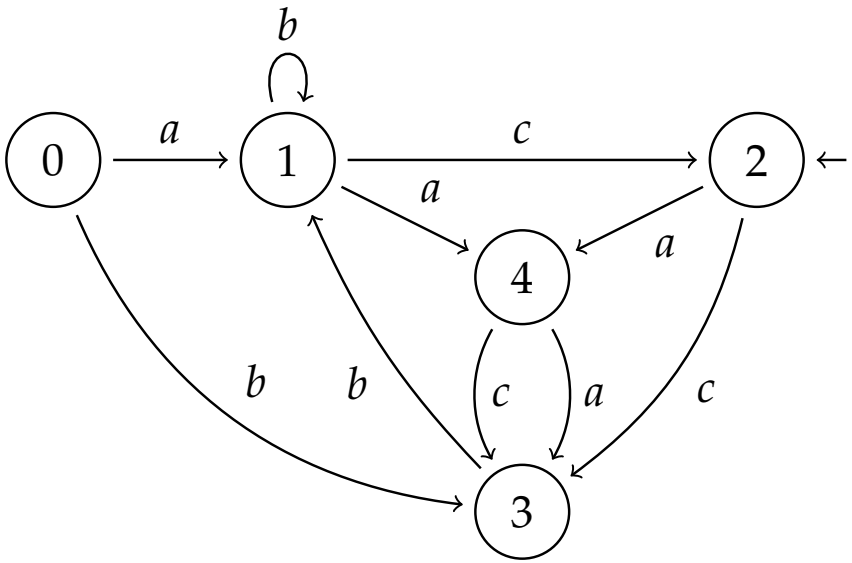
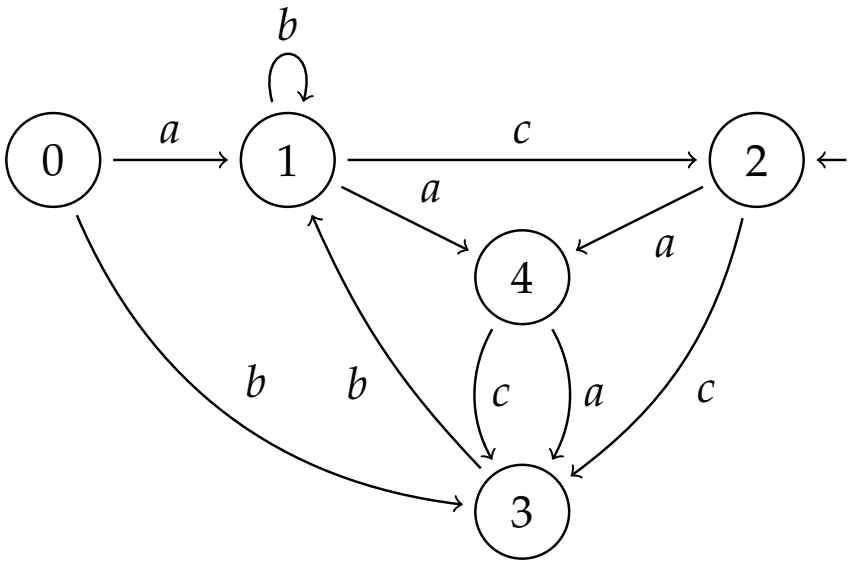


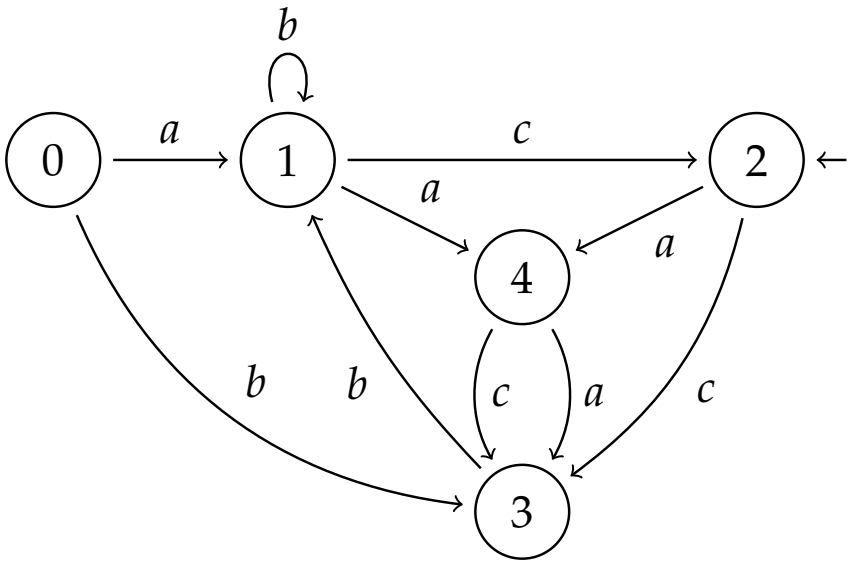
1. {2}

FCIter in action



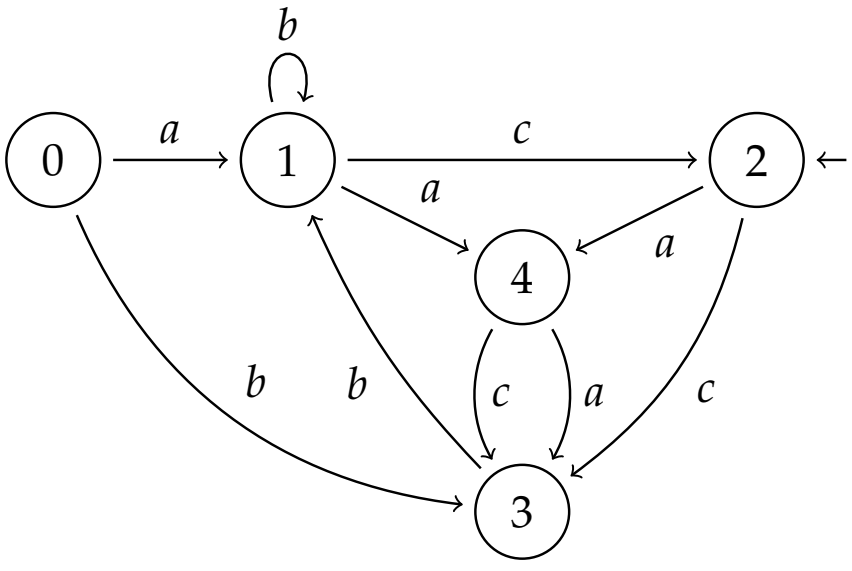
1. {2}
2. {2,3}

FCIter in action



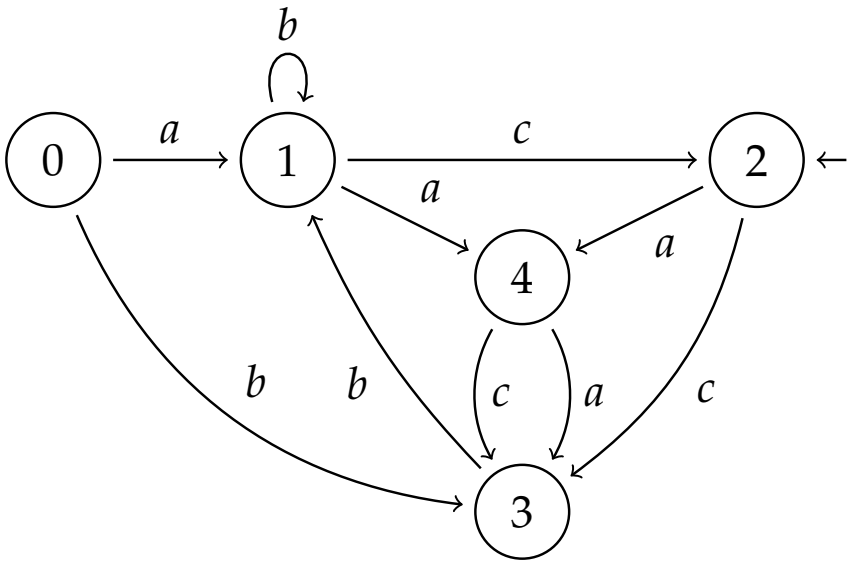
1. {2}
2. {2,3}
3. {2,3,4}

FCIter in action



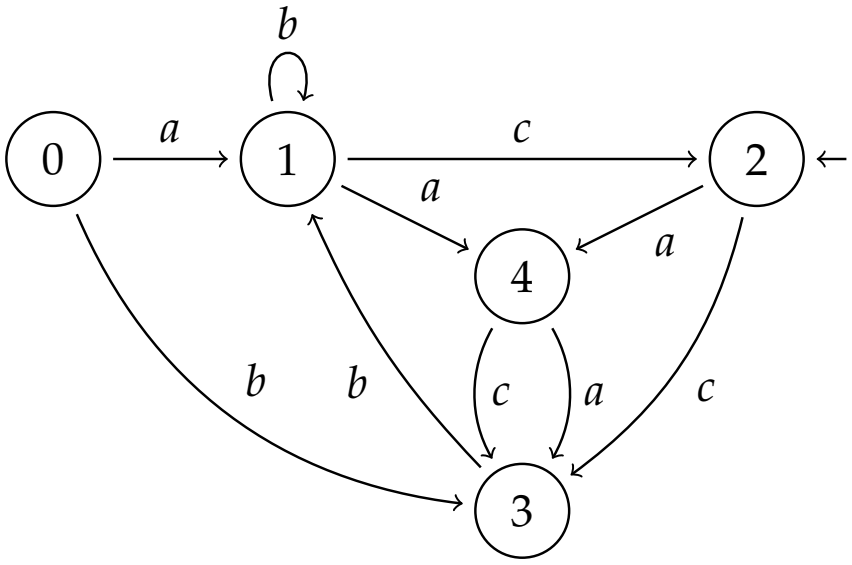
1. {2}
2. {2,3}
3. {2,3,4}
4. {2,3,4,1}

FCIter in action



1. {2}
2. {2,3}
3. {2,3,4}
4. {2,3,4,1}
5. {2,3,4,1} ← fixed point

FCIter in action



1. {2}
2. {2,3}
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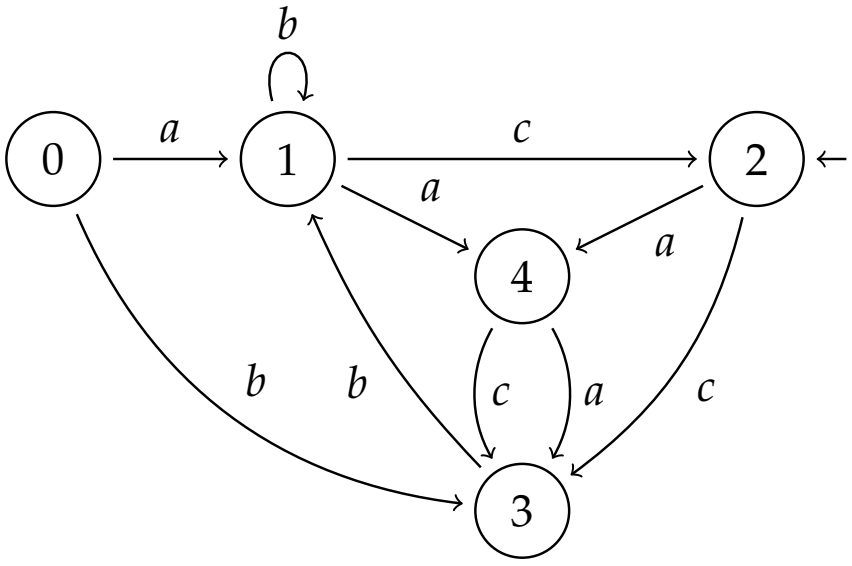
Allows to decide

- language is Σ^*.

FCIter in action



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Allows to decide

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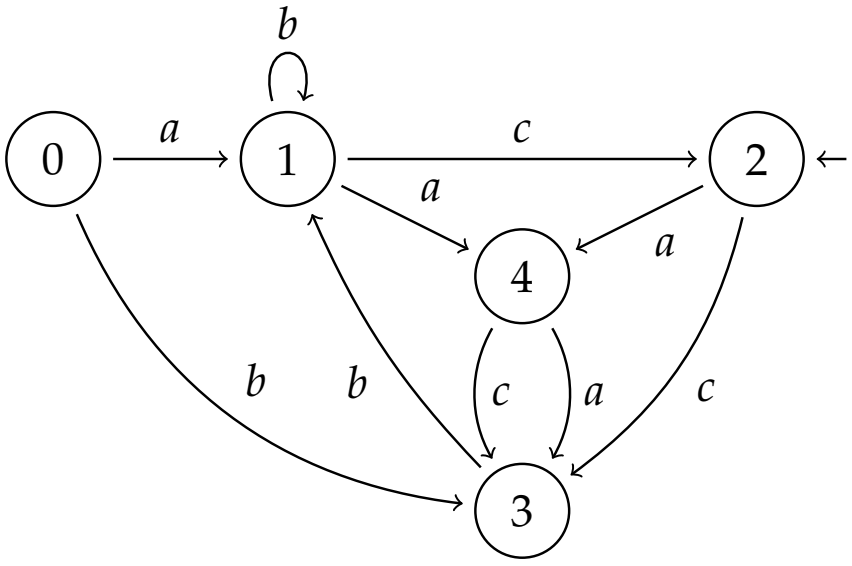
FCIter in action



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3. {2,3,4}
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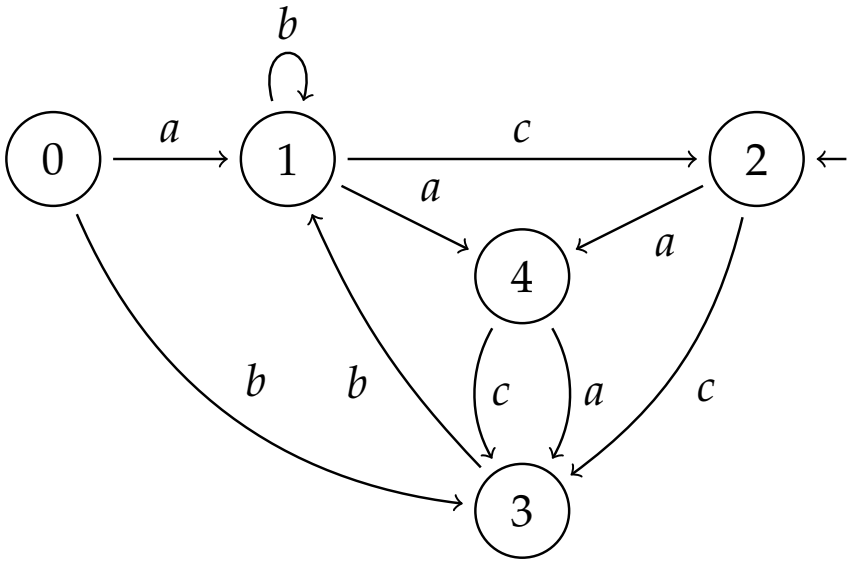
5. {2,3,4,1} ← fixed point

Allows to decide

- language is Σ^*.
- language emptiness.
- language inclusion.

FCIter in action



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3. {2,3,4}
4. {2,3,4,1}
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Allows to decide

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- language emptiness.
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- language equivalence.

More about automata

- Closure properties

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- Closure properties
  - Complement

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  - Complement
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- Closure properties
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The End

Thank you for your attention

Any questions? Ask away!

Extras

Definition (pure)

For a decidable predicate p with

`pure p x:= if p x then ⊤ else ⊥`

pure p is a *pure* predicate.

Reachable states in DFAs

Let A be a DFA with set of states S

We use finite closure iteration to compute reachable states.

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Definition (step predicate)

```
step_reach (set: list (S A)) (q : S A):=
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$\exists\, q'\ x,\ q' \in set \to \delta_S\ q'\ x = q.$

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Definition

reach `reach (q:S) := FCIter step_reach [q].`

eqTypes

```
dec (P:ℙ) := {P} + {¬ P}.

eq_dec (X:Type ) := ∀ x  y, dec (x = y).

Structure eqType := EqType {
eqtype :> Type ;
decide_eq :  eq_dec eqtype }.
```

finTypes

```
Class finTypeC (type: eqType): Type := FinTypeC {
  enum: list type;
  enum_ok: ∀ x: type, count enum x = 1 }.

Structure finType: Type := FinType {
 type :> eqType;
 class : finTypeC type }.
```

Admissible functions

Definition (Admissibility)

A function (f: list Y $\to$ list Y) is called *admissible* if a given list A is either a fixed-point of f or $card\ (f\ A) > card\ A$.

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Theorem

Let f be an admissible function $list\ X \to list\ X$. Then

$$f^{Cardinality\ X}\ A$$

is a fixed point of f for any list A over elements of X.

FCIter

Lemma

FCStep is an admissible function.

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`FCIter := FCStep^Cardinality X.`

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Lemma

FCStep is an admissible function.

FCIter

`FCIter := FCStep`$^{\texttt{Cardinality X}}$.

Corollary

Let A be a list over X. Then `FCIter A` is a fixed point of *FCStep*.

A FINITE VECTOR TYPE

Goal: Construct finite type for $X_1 \longrightarrow X_2$.

A finite vector type

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Needed: List containing all vectors of type $X_1 \longrightarrow X_2$.

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```
Fixpoint images (Y: Type ) (A: list Y) (n:  ℕ) :  list (list Y)
:=
match n with
| 0 ⇒ [[]]
| S n' ⇒ concat (map (λ x ⇒  map (cons x) (images A n')) A)
end.
```

A FINITE VECTOR TYPE

Goal: Construct finite type for $X_1 \longrightarrow X_2$.

Needed: List containing all vectors of type $X_1 \longrightarrow X_2$.

Fact

$\forall\, A, A \in \mathit{images}\ (\mathit{elem}\ X_2)\ (\mathit{Cardinality}\ X_1) \rightarrow |A| = \mathit{Cardinality}\ X_1.$

$\Rightarrow$ We can build vectors

Fact

Let A: *list* X_2 and $|A| = \mathit{Cardinality}\ X$. Then
$\mathit{count}\ (\mathit{images}\ (\mathit{elem}\ X_2)\ (\mathit{Cardinality}\ X_1))\ A = 1.$

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This is enough to construct finite type $X_2^{X_1}$.