Formalizing Strong Representability Theorems for Gödel's First Incompleteness Theorem and Other Applications

Benjamin Peters
Universität des Saarlandes

March 26, 2022

There is a simple folklore proof of Gödel's first incompleteness theorem (G1) by Kleene using computability theory and undecidability of the halting problem [8]. It shows incompleteness of formal systems that weakly represent the halting problem, such as first-order logic with the axioms of Robinson's Q. Kleene's proof is much easier to spell out in detail than the original Gödel-Rosser proof [5, 15], only relying on basic results in computability theory [10].

However, Kleene's well-known result is weaker than the Gödel-Rosser proof: It only works for sound as opposed to just consistent formal systems, and does not construct an independent sentence. Similarly, Gödel's original result only applied to omega-consistent theories until it was strengthened by Rosser to only require consistency. Kleene also found a way to fix these weaknesses in his proof, in part using the same trick Rosser used [9]. We described an abstract presentation of these results in [13], assuming a form of representability we called value representability.

In this memo, we show how to apply Rosser's trick to weak representability to obtain strong representability theorems, such as strong separability of disjoint predicates (as done by Kleene [9]), value representability, Church's thesis for Q (CT_Q) assuming Church's thesis for a concrete machine model (CT_L), and more.

1 Preliminaries

We work with a presentation of intuitionistic first-order logic with the theories of (intuitionistic) Robinson's Q as well as Heyting arithmetic HA, and the standard model of natural numbers $\mathbb{N}$, as presented in [7]. All theorems can also be derived for their classical counterparts PQ and PA.

Definition 1 (Comparison operators)**.** *We define the following derived notion on formulas:*

$$x \le y \;:=\; \exists z.\, y = x + z \lor y = z + x$$

In HA, $x \le y$ can easily be shown equivalent to the more conventional $\exists z.\, y = x + z$ using commutativity of addition. This does not hold in Q.

Definition 2 (Δ_1 formulas)**.** *A formula φ is Δ_1, if for any closed substitution ρ, $\varphi[\rho]$ is Q-decidable, that is $\mathsf{Q} \vdash \varphi$ or $\mathsf{Q} \vdash \neg\varphi$.*

Lemma 3. *The following formulas are Δ_1:*

1. *propositional formulas (including falsity and equations),*
2. *comparisons $x \le y$,*
3. *bounded quantifiers $\forall x \le t.\, \varphi$ or $\exists x \le t.\, \varphi$, where t is a term that does not contain x,*
4. *binary bounded quantifiers $\forall xy.\, x + y \le t \;\rightarrow\; \varphi$, where t is a term that does not contain x and y.*

Proof. The proofs of *1.* and *2.* are easy. Bounded quantification can be shown to be equivalent to finite conjunction/disjunction. □

Definition 4 (Σ_1 and Π_1 formulas)**.** *We say that a formula is Σ_1 if it is of the form $\exists m_1, m_2, \ldots, m_n.\, \psi$ where ψ is Δ_1. We say that a formula is Π_1 if it is of the form $\forall m_1, m_2, \ldots, m_n.\, \psi$ where ψ is Δ_1.*

Lemma 5 ($\exists$ compression)**.** *For any formula $\varphi \in \Sigma_1$ there is a formula $\psi \in \Delta_1$ such that*

$$\mathsf{Q} \vdash \varphi \;\leftrightarrow\; \exists m.\, \psi.$$

Proof. It suffices to show that we can compress two existential quantifiers, that is, for any $\varphi \in \Delta_1$:

$$\exists \psi \in \Delta_1.\, \mathsf{Q} \vdash (\exists xy.\, \varphi(x, y)) \;\leftrightarrow\; \exists z.\, \psi(z)$$

Choose

$$\psi(z) := \exists x \le z.\, \exists y \le z.\, \varphi(x, y)$$

The rest of this proof is done formally in Q. The direction from right to left is trivial. Let x, y be such that $\varphi(x, y)$. Choose $z := x + y$. Both bounds can easily be shown since our definition of $\le$ accommodates the absence of commutativity. □

We do not know of a way to show Lemma 5 in Q using only a simpler definition of $\le$.

Definition 6 (Weak representability)**.** *A formula $\varphi \in \Sigma_1$ with a single free variable weakly represents a predicate $P : \mathbb{N} \to \mathbb{P}$ if for all x:*

$$Px \;\leftrightarrow\; \mathsf{Q} \vdash \varphi(\overline{x}).$$

Remark 7. *It was shown by [12, 7] that all predicates enumerable in a concrete model of computation* L *[4] are weakly Σ_1-representable, that is, weakly representable by a Σ_1-formula. By assuming* CT_{L} *[11, 16, 2] this also applies to all synthetically [14, 1, 3] enumerable predicates.*

Definition 8 (Strong separability)**.** *A formula φ with a single free variable strongly separates two disjoint predicates A_1, A_2 if for all x:*

$$x \in A_1 \;\rightarrow\; Q \vdash \varphi(x) \qquad\qquad x \in A_2 \;\rightarrow\; Q \vdash \neg\varphi(x)$$

The definitions of representability and separability can easily be extended to predicates of arbitrary arity.

Lemma 9 (Σ_1-completeness)**.** *Let $\varphi \in \Sigma_1$ be a closed formula. Then $\mathbb{N} \vDash \varphi \;\rightarrow\; \mathsf{Q} \vdash \varphi$.*

Proof. Let $\varphi = \exists m_1, \ldots, m_n.\, \psi$ be with $\psi \in \Delta_1$. We obtain $m_1, \ldots, m_n \in \mathbb{N}$ and $\mathbb{N} \vDash \psi(m_1, \ldots, m_n)$. By decidability (Lemma 3) and soundness, $\mathsf{Q} \vdash \psi(m_1, \ldots, m_n)$ must hold. □

Corollary 10 (Σ_1-witnesses)**.** *Witnesses for closed Σ_1-formulas are always standard, that is, for any formula $\varphi \in \Sigma_1$ with a single free variable x:*

$$\mathsf{Q} \vdash \exists x.\, \varphi(x) \;\rightarrow\; \exists x.\, \mathsf{Q} \vdash \varphi(\overline{x}).$$

Proof. By extracting a witness in $\mathbb{N}$ using soundness and reestablishing the formula using Σ_1-completeness. □

1.1 Rosser's trick

Gödel's proof of G1 relies on the arithmetization of provability in the form of a binary provability relation Prf_F such that

$$F \vdash \varphi \;\leftrightarrow\; F \vdash \exists k.\, \mathrm{Prf}_F(\overline{\ulcorner\varphi\urcorner}, k),$$

from which the independent Gödel sentence is constructed. Here, F denotes an arbitrary enumerable and ω-consistent theory that subsumes Q and $\ulcorner\cdot\urcorner$ denotes some Gödelization of formulas. Rosser defined a modified provability relation Prf'_F:

$$\mathrm{Prf}'_F(x, k) := \mathrm{Prf}_F(x, k) \wedge \forall k' \leq k.\, \neg\mathrm{Prf}_F(\mathrm{neg}(x), k),$$

where $\mathrm{neg} : \mathbb{N} \rightarrow \mathbb{N}$ negates a gödelized formula. Intuitively, $\exists k.\, \mathrm{Prf}'_F(\overline{\ulcorner\varphi\urcorner}, k)$ states that there is a proof of φ and there is no smaller refutation of φ. Rosser showed that $\exists k.\, \mathrm{Prf}'_F(x, k)$ strongly separates the sets of provable and refutable formulas, which allowed him to weaken the requirement of ω-consistency for F, leaving only consistency and enumerability.

2 Strong separability

Lemma 11 (Decidability of $\leq$)**.** *Let $x \in \mathbb{N}$. Then*

$$\mathsf{Q} \vdash \forall y.\, \overline{x} \leq y \vee y \leq \overline{x}$$

Proof. By induction on x and a case distinction on y in the successor case. $\square$

We would not be able to show this if x was quantified within the formula, since we would not be able to do induction, as Q does not have an induction scheme.

Theorem 12 (Strong separability of disjoint predicates)**.** *Let P_1, P_2 be disjoint and weakly Σ_1-representable predicates. There are formulas $\varphi_1, \varphi_2 \in \Sigma_1$ that both strongly separate P_1, P_2, that is:*

$$P_1 x \;\rightarrow\; \mathsf{Q} \vdash \varphi_1(\overline{x}) \qquad (1) \qquad\qquad P_1 x \;\rightarrow\; \mathsf{Q} \vdash \neg\varphi_2(\overline{x}) \qquad (3)$$
$$P_2 x \;\rightarrow\; \mathsf{Q} \vdash \varphi_2(\overline{x}) \qquad (2) \qquad\qquad P_2 x \;\rightarrow\; \mathsf{Q} \vdash \neg\varphi_1(\overline{x}) \qquad (4)$$

Proof. Using Lemma 5, let $\psi_1, \psi_2 \in \Delta_1$ be such that

$$\forall x.\, P_1 x \;\leftrightarrow\; \mathsf{Q} \vdash \exists k.\, \psi_1(\overline{x}, k) \qquad (5) \qquad\qquad \forall x.\, P_2 x \;\leftrightarrow\; \mathsf{Q} \vdash \exists k.\, \psi_2(\overline{x}, k) \qquad (6)$$

Choose

$$\Phi_1(x, k) := \psi_1(x, k) \wedge \forall k' \leq k.\, \neg\psi_2(x, k')$$
$$\Phi_2(x, k) := \psi_2(x, k) \wedge \forall k' \leq k.\, \neg\psi_1(x, k')$$

Now, $\varphi_1(x) := \exists k.\, \Phi_1(x, k)$ and $\varphi_2(x) := \exists k.\, \Phi_2(x, k)$ fulfil (1) through (4):

(1) Let $x \in \mathbb{N}$ be such that $P_1 x$. By (5) and soundness we have a $k \in \mathbb{N}$ such that $\mathbb{N} \vDash \psi_1(\overline{x}, \overline{k})$. By Σ_1-completeness it suffices to show $\mathbb{N} \vDash \exists k.\, \Phi_1(\overline{x}, k)$. By choosing k, the first conjunct is trivial. For the second one, let $k' \leq k$ be such that $\mathbb{N} \vDash \psi_2(\overline{x}, \overline{k'})$. By Σ_1-completeness and (5) we have $P_2 x$, which contradicts disjointness.

(2) Analogous to (1).

(3) Let $x \in \mathbb{N}$ be such that $P_1 x$. By (1) we have $Q \vdash \varphi_1(\overline{x})$ and by Corollary 10 we have a $k_1 \in \mathbb{N}$ such that $Q \vdash \psi_1(\overline{x}, \overline{k_1}) \wedge \forall k_1' \leq \overline{k_1}.\, \neg\psi_2(\overline{x}, k_1')$. The rest of this proof is done formally in Q. Assume a k_2 such that $\psi_2(\overline{x}, k_2)$ and $\forall k_2' \leq k_2.\, \neg\psi_1(\overline{x}, k_2')$. We are done by doing a case distinction on whether $\overline{k_1} \leq k_2$ or $k_2 \leq \overline{k_1}$ using Lemma 11 and instantiating one of the quantified assumptions.

(4) Analogous to (3). $\square$

By weakening all consistent extensions of Q also strongly separate such predicates, even if they are unsound. The definitions of Φ_1 and Φ_2 are an application of Rosser's trick close to Rosser's original use.

Theorem 12 can be instantiated to yield value representability for partial functions $\mathbb{N} \rightharpoonup \mathbb{B}$ as described in [13]. This allows us to show Gödel's first incompleteness theorem only assuming CT_L.

Corollary 13 (Decidable predicates)**.** *Let P be a predicate such that both P and $\overline{P}$ are weakly Σ_1-representable. There are formulas $\Phi_1 \in \Sigma_1, \Phi_2 \in \Pi_1$ that both strongly represent P and $\overline{P}$, that is*

$$Px \rightarrow \mathsf{Q} \vdash \Phi_1(\overline{x}) \qquad\qquad Px \rightarrow \mathsf{Q} \vdash \neg\Phi_2(\overline{x})$$
$$\neg Px \rightarrow \mathsf{Q} \vdash \neg\Phi_1(\overline{x}) \qquad\qquad \neg Px \rightarrow \mathsf{Q} \vdash \Phi_2(\overline{x})$$

Proof. By Theorem 12, let $\varphi_1, \varphi_2 \in \Sigma_1$ be predicates that separate the disjoint sets P and $\overline{P}$. Choose $\Phi_1 := \varphi_1$. It is easy to find a $\Phi_2 \in \Pi_1$ that is equivalent to $\neg\varphi_2$. ☐

By Remark 7 and assuming CT_L, such predicates are for instance the synthetically decidable predicates, since they are both synthetically enumerable and co-enumerable.

Corollary 14 (Deep disjointness)**.** *Let P_1, P_2 be disjoint and weakly Σ_1-representable predicates and φ_1, φ_2 be chosen as in the proof of Theorem 12. Now,*

$$\mathsf{PA} \vdash \forall x.\, \neg(\varphi_1(x) \land \varphi_2(x)).$$

Proof. This proof is done formally in PA. As opposed to in Q, it is possible to show $\mathsf{PA} \vdash \forall xy.\, x \leq y \lor y \leq x$ by induction on x. Now, obtain the witnesses from $\varphi_1(x)$ and $\varphi_2(x)$, compare them and instantiate one of the assumptions. ☐

We do not know of a way to show this in Q. A related property was assumed in [6].

3 Functions

Theorem 15 (CT_Q)**.** *Let $f : \mathbb{N} \rightharpoonup \mathbb{N}$ be a partial function such that the graph of f, that is $\{(x, y) \mid fx \rhd y\}$, is weakly Σ_1-representable. There is a $\varphi \in \Sigma_1$ such that*

$$fx \rhd y \rightarrow Q \vdash \forall y'.\, \varphi(\overline{x}, y') \leftrightarrow y' = \overline{y}$$

Proof. By Lemma 5, let $\psi \in \Delta_1$ be such that

$$fx \rhd y \leftrightarrow Q \vdash \exists k.\, \psi(x, y, k).$$

Choose

$$\Phi(x, y, k) := \psi(x, y, k) \land \forall y'k'.\, y' + k' \leq y + k \rightarrow \psi(x, y', k') \rightarrow y' = y$$
$$\varphi(x, y) := \exists k.\, \Phi(x, y, k).$$

Assume $fx \rhd y$. The proof of $\mathsf{Q} \vdash y' = y \;\rightarrow\; \forall y'.\, \varphi(\overline{x}, y')$ is similar to the proof of (1) above. The rest of this proof is done formally in Q, except when stated otherwise.

Assume y', k' such that $\Phi(\overline{x}, y', k')$. By $fx \rhd y$ and the direction from right to left we also have $\varphi(\overline{x}, \overline{y})$ and therefore by Corollary 10 a $k \in \mathbb{N}$ such that $\Phi(\overline{x}, \overline{y}, \overline{k})$. We are done by doing a case distinction on whether $\overline{y + k} \leq y' + k'$ or $y' + k' \leq \overline{y + k}$ using Lemma 11. □

The graph of a partial function is synthetically enumerable. Using Remark 7 we can therefore deduce $\mathsf{CT_Q}$ for all partial functions only assuming CT_{L}.

Corollary 16 ($\mathsf{CT_Q}$ for total functions)**.** *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a (total) function. Assuming CT_{L}, there is a $\varphi \in \Sigma_1$ such that*

$$\mathsf{Q} \vdash \forall y.\, \varphi(\overline{x}, y') \;\leftrightarrow\; y = \overline{fx}.$$

This version of $\mathsf{CT_Q}$ restricted to partial functions was assumed in [6].

Corollary 17 (Value representability)**.** *Let $f : \mathbb{N} \rightharpoonup \mathbb{N}$ be a partial function. Assuming CT_{L}, there is a $\varphi \in \Sigma_1$ that value-represents f, that is:*

$$fx \rhd y \;\rightarrow\; \mathsf{Q} \vdash \varphi(\overline{x}, \overline{y}) \;\wedge\; \forall y' \neq y.\, \mathsf{Q} \vdash \neg\varphi(\overline{x}, \overline{y'})$$

Value representability appears to be weaker than $\mathsf{CT_Q}$ because we lose information on the behavior of $\varphi(\overline{x}, y)$ for non-standard y.

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