Formalizing Strong Representability Theorems for Gödel's First Incompleteness Theorem and Other Applications

Second Bachelor's Seminar Talk

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Computational Folklore Proof

Theorem (Folklore G1)[1]

Any axiomatization T that weakly represents the halting problem

$$\lambda x.\, \exists y.\, \theta x x \triangleright y,$$

that is, there is a formula φ such that

$$\forall x.\, (\exists y.\, \theta x x \triangleright y) \;\leftrightarrow\; T \vdash \varphi(x),$$

is incomplete.

Folklore G1 for Robinson's Q

$\forall T \supseteq \mathsf{Q}.\, \mathbb{N} \vDash T \;\rightarrow$	soundness
T enumerable $\;\rightarrow$	effectiveness
$\neg(\forall \varphi.\, T \vdash \varphi \vee T \vdash \neg\varphi)$	incompleteness

[1] Kleene 1952.

Church's Thesis

Assumption ($\mathsf{CT_L}$)[2]

The interpreter $\theta : \mathbb{N} \to \mathbb{N} \rightharpoonup \mathbb{N}$ for L[3] satisfies

$$\forall (f : \mathbb{N} \rightharpoonup \mathbb{N}).\, \exists c.\, \forall xy.\, fx \triangleright y \;\leftrightarrow\; \theta c x \triangleright y.$$

Corollary

Let $P : \mathbb{N} \to \mathbb{P}$ be a predicate. It is equivalent:

- P is synthetically enumerable:

$$\exists f : \mathbb{N} \to \mathcal{O}(\mathbb{N}).\, \forall x.\, Px \;\leftrightarrow\; \exists k.\, fk = {}^{\circ}x$$

- P is enumerable in L:

$$\exists c : \mathbb{N}.\, \forall x.\, Px \;\leftrightarrow\; \exists k.\, \theta c k \triangleright x$$

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[2] Forster 2021.

[3] a Turing-complete λ-calculus (Forster and Smolka 2017).

Weak Representability in Q

Theorem (Weak representability)[4]

Every predicate $P : \mathbb{N} \to \mathbb{P}$ enumerable in L can be weakly Σ_1-represented in Q, that is, there is a formula $\varphi \in \Delta_1$ such that:

$$\forall x.\, Px \;\leftrightarrow\; \mathsf{Q} \vdash \exists k.\, \varphi(x, k)$$

[4] largely mechanized by Larchey-Wendling and Forster 2019; Kirst and Hermes 2021.

Strengthened Computational Proof

Theorem (Strengthened G1)[5]

In any axiomatization T that strongly separates

$$\lambda x.\, \theta x x \triangleright 1 \qquad\qquad \lambda x.\, \theta x x \triangleright 0,$$

that is, there is a formula φ such that for all x

$$\theta x x \triangleright 1 \;\rightarrow\; T \vdash \varphi(x) \qquad\qquad \theta x x \triangleright 0 \;\rightarrow\; T \vdash \neg\varphi(x),$$

there is an independent sentence ψ, that is $T \nvdash \psi$ and $T \nvdash \neg\psi$.

Strengthened G1 for Robinson's Q

$\forall T \supseteq \mathsf{Q}.\, T \nvdash \bot \;\rightarrow$	consistency
T enumerable $\rightarrow$	effectiveness
$\exists \psi.\, T \nvdash \psi \wedge T \nvdash \neg\psi$	independent sentence

[5]Kleene 1952.

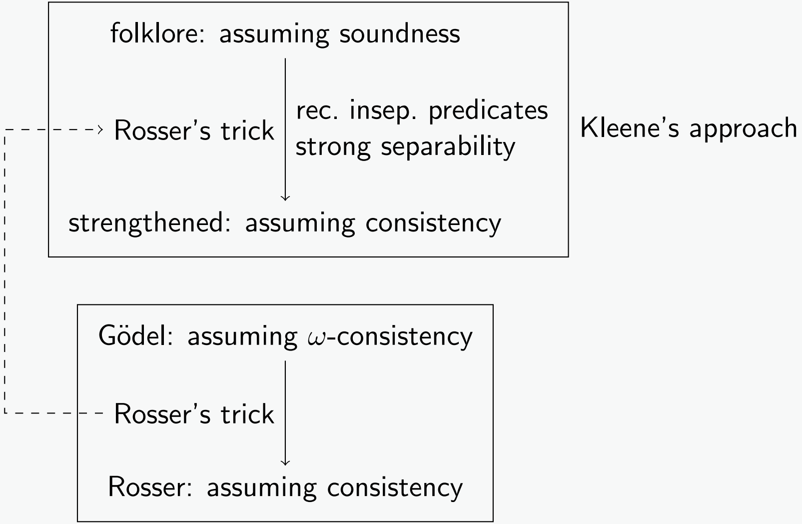
Strong Separability in Q

Theorem (Strong separability)

Let P_1, P_2 be disjoint and enumerable predicates.
In Q, P_1 and P_2 are strongly separable, that is, there is a formula Φ such that for all x:

$$P_1 x \;\rightarrow\; \mathsf{Q} \vdash \Phi(x) \qquad\qquad P_2 x \;\rightarrow\; \mathsf{Q} \vdash \neg\Phi(x)$$

Path Towards Strengthened G1



Original Proof

Gödel's approach[6]

Construct a formula $\mathrm{Prf} \in \Delta_1$ weakly representing provability:

$$\forall \varphi.\, \mathsf{Q} \vdash \varphi \;\leftrightarrow\; \mathsf{Q} \vdash \exists k.\, \mathrm{Prf}(\ulcorner \varphi \urcorner, k)$$

Rosser's trick[7]

Modify provability predicate such that for all φ:

$$\mathsf{Q} \vdash \varphi \;\rightarrow\; \mathsf{Q} \vdash \exists k.\, \mathrm{Prf}'(\ulcorner \varphi \urcorner, k)$$
$$\mathsf{Q} \vdash \neg\varphi \;\rightarrow\; \mathsf{Q} \vdash \neg\exists k.\, \mathrm{Prf}'(\ulcorner \varphi \urcorner, k)$$

[6] Gödel 1931.
[7] Rosser 1936.

Rosser's Trick for Strong Separability

Let P_1, P_2 be disjoint predicates, $\varphi_1, \varphi_2 \in \Delta_1$ such that:

$$P_1 x \;\leftrightarrow\; \mathsf{Q} \vdash \exists k.\, \varphi_1(x, k)$$
$$P_2 x \;\leftrightarrow\; \mathsf{Q} \vdash \exists l.\, \varphi_2(x, l)$$

We want to find Φ_1, Φ_2 such that for all x:

$$P_1 x \;\rightarrow\; \mathsf{Q} \vdash \exists k.\, \Phi_1(x, k) \qquad P_2 x \;\rightarrow\; \mathsf{Q} \vdash \neg\exists k.\, \Phi_1(x, k)$$
$$P_2 x \;\rightarrow\; \mathsf{Q} \vdash \exists l.\, \Phi_2(x, l)$$

To do this, choose:

$$\Phi_1(x, k) := \varphi_1(x, k) \land \forall k' \leq k.\, \neg\varphi_2(x, k')$$
$$\Phi_2(x, l) := \varphi_2(x, l) \land \forall l' \leq l.\, \neg\varphi_1(x, l')$$

Key property:

$$\neg(\Phi_1(x, k) \land \Phi_2(x, l))$$

Strong Separability in Q

Theorem (Strong separability)

Let P_1, P_2 be disjoint and weakly Σ_1-representable predicates. Now, P_1 and P_2 are strongly Σ_1-separable, that is, there is a formula $\Phi \in \Sigma_1$ such that for all x:

$$P_1 x \;\rightarrow\; \mathsf{Q} \vdash \Phi(x) \qquad\qquad P_2 x \;\rightarrow\; \mathsf{Q} \vdash \neg\Phi(x)$$

Corollary (Strong representability)

In Q, a decidable predicates P is strongly representable, that is, there is a formula $\Phi \in \Sigma_1$ (or $\Phi \in \Pi_1$) such that for all x:

$$Px \;\to\; \mathsf{Q} \vdash \Phi(x) \qquad\qquad \neg Px \;\to\; \mathsf{Q} \vdash \neg\Phi(x)$$

Theorem (Deep disjointness)[8]

Let P_1, P_2 be disjoint and enumerable predicates.
There are $\Phi_1, \Phi_2 \in \Sigma_1$ that weakly represent and strongly separate P_1, P_2 and:

$$\mathsf{PA} \vdash \forall x.\, \neg(\Phi_1(x) \wedge \Phi_2(x))$$

Theorem ($\mathsf{CT_Q}$)

Let $f : \mathbb{N} \rightharpoonup \mathbb{N}$ be a partial function. There is a $\Phi \in \Sigma_1$ such that

$$\forall xy.\, fx \triangleright y \;\to\; \mathsf{Q} \vdash \forall y'.\, \Phi(x, y') \;\leftrightarrow\; y' = y$$

[8] Hermes and Kirst N.D.

Conclusion

I showed, using ideas from Kleene 1952:

1. essential incompleteness for abstract formal systems that strongly separate certain predicates using synthetic computability theory.
2. that Robinson's Q strongly separates disjoint and enumerable predicates.
3. that Robinson's Q fulfills other representability properties.

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Proof of $\mathsf{CT_Q}$

Theorem ($\mathsf{CT_Q}$)

Let $f : \mathbb{N} \rightharpoonup \mathbb{N}$ be a partial function. There is a $\Phi \in \Sigma_1$ such that

$$\forall xy.\, fx \triangleright y \;\rightarrow\; \mathsf{Q} \vdash \forall y'.\, \Phi(x, y') \;\leftrightarrow\; y' = y$$

Proof.

Let $\varphi \in \Sigma_1$ be such that:

$$fx \triangleright y \;\leftrightarrow\; \mathsf{Q} \vdash \exists k.\, \varphi(x, y, k)$$

Choose:

$$\begin{aligned} \Phi(x, y) := \exists k. \varphi(x, y, k) \wedge \\ \forall y' k'.\, y' + k' \leq y + k \;\rightarrow\; \varphi(x, y', k') \;\rightarrow\; y' = y \end{aligned}$$

□

Δ_1

Definition (Δ_1)

A formula φ is Δ_1, if for any valuation ρ:

$$\mathsf{Q} \vdash \varphi[\rho] \lor \mathsf{Q} \vdash \neg\varphi[\rho]$$

Lemma (Δ_1)

All formulas only containing bounded quantifiers are Δ_1.

∃-compression

$$x \leq y := \exists z.\, x + z = y \vee z + x \leq y$$

Let $\varphi \in \Sigma_1$.

$$Q \vdash \exists x.\, \exists y.\, \varphi(x, y) \;\leftrightarrow\; \exists z.\, \exists x \leq z.\, \exists y \leq z.\, \varphi(x, y)$$

Proof: By choosing $z := x + y$.