Gödel's Theorem Without Tears

Essential Incompleteness in Synthetic Computability

Final Bachelor Talk

Benjamin Peters

Advisor: Dominik Kirst
Supervisor: Professor Gert Smolka

Saarland University

17th June, 2022

Gödel's Theorem Without Tears[1]

Essential Incompleteness in Synthetic Computability

22nd June, 2022
TYPES 2022

Benjamin Peters Dominik Kirst

SAARLAND UNIVERSITY
COMPUTER SCIENCE

[1] Abstract title: "Strong, Synthetic, and Computational Proofs of Gödel's First Incompleteness Theorem"

Gödel's First Incompleteness Theorem

Gödel's first incompleteness theorem[2]

Any effective, sound, and sufficiently powerful formal logic is incomplete.

- Has been mechanised often[3]
- We present Kleene's folklore and strengthened incompleteness proofs using computability theory *abstractly*
- We formalise them in the setting of *synthetic computability theory*, avoiding low-level manipulations
- We instantiate these results to first-order Robinson arithmetic
- All results have been mechanised in Coq[4]

---

[2] Gödel 1931.
[3] Shankar 1994; O'Connor 2005; Harrison 2009; Paulson 2014; Popescu and Traytel 2019.
[4] `https://github.com/uds-psl/coq-synthetic-incompleteness/tree/bachelor`

Abstract Incompleteness Proofs

Instantiation to first-order Robinson arithmetic

Synthetic Computability[5]

We work in CIC, where we can consider the function space to only contain computable functions

Definition

A predicate $P : X \to \mathbb{P}$ is

- enumerable if $\exists f : \mathbb{N} \to \mathcal{O}(X).\, Px \;\leftrightarrow\; \exists k.\, fk = \ulcorner x \urcorner$.
- decidable if $\exists f : X \to \mathbb{B}.\, Px \;\leftrightarrow\; fx = \mathit{tt}$.

[5] Richman 1983; Bauer 2006.

Formal Systems

Definition (Formal system)

$\mathcal{F} = (S, \neg, \vdash)$ is a formal system if:

- $S : \mathbb{T}$ is a discrete type of sentences
- $\neg : S \to S$ is a negation function
- $\vdash : S \to \mathbb{P}$ is an enumerable provability predicate
- $\mathcal{F}$ is consistent: $\forall s.\, \neg(\mathcal{F} \vdash s \wedge \mathcal{F} \vdash \neg s)$

$\mathcal{F}$ is complete if $\forall s.\, \mathcal{F} \vdash s \vee \mathcal{F} \vdash \neg s$.

First-order logic over a consistent and enumerable axiomatisation is a formal system in this sense

Decidable Formal Systems

Lemma

There is a partial function $d_{\mathcal{F}} : S \rightharpoonup \mathbb{B}$ separating provability from refutability:

$$\forall s.\,(d_{\mathcal{F}}\,s \triangleright \mathit{tt} \;\leftrightarrow\; \mathcal{F} \vdash s) \land (d_{\mathcal{F}}\,s \triangleright \mathit{ff} \;\leftrightarrow\; \mathcal{F} \vdash \neg s)$$

If $\mathcal{F}$ is complete, $d_{\mathcal{F}}$ is total.

Corollary

Any complete formal system is decidable.

Kleene's Folklore Incompleteness Proof[6,7]

Theorem

Let $\mathcal{F}$ be complete and weakly represent $P : \mathbb{N} \to \mathbb{P}$, i.e., there is an $r : \mathbb{N} \to S$ s.t.:

$$\forall x.\, Px \;\leftrightarrow\; \mathcal{F} \vdash rx$$

Then P is decidable. Thus, if P is undecidable, $\mathcal{F}$ is incomplete.

[6] Kleene 1936; Turing 1936.

[7] As mechanised by Kirst and Hermes 2021.

Church's Thesis[9]

Axiom (EPF[8])

There is a function $\theta : \mathbb{N} \to \mathbb{N} \rightharpoonup \mathbb{B}$ such that:

$$\forall f : \mathbb{N} \rightharpoonup \mathbb{B}.\, \exists c.\, f \equiv \theta c$$

Definition (Self-halting problem)

The self-halting problem is defined as:

$$\mathcal{H} := \lambda x.\, \exists b.\, \theta x x \triangleright b$$

[8] Richman 1983; Forster 2022.
[9] Kreisel 1967; Troelstra and van Dalen 1988.

Self-halting problem

Fact

Partial functions $f : \mathbb{N} \rightharpoonup \mathbb{B}$ agreeing with the halting problem $\mathcal{H} := \lambda x.\, \exists b.\, \theta x x \triangleright b$:

$$\forall x.\, x \in \mathcal{H} \;\leftrightarrow\; f x \triangleright \textit{tt},$$

diverge on some input c, i.e., $\forall b.\, f c \not\triangleright b$.

Proof.

Consider $g : \mathbb{N} \rightharpoonup \mathbb{B}$,

$$g x := \begin{cases} \textit{ff} & \text{if } f x \triangleright \textit{tt} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Let c be the code of g. We have $f c \triangleright \textit{tt} \;\leftrightarrow\; f c \triangleright \textit{ff}$. □

Strengthening the Folklore Proof[10]

Theorem

Assume $\mathcal{F}$ weakly represents $\mathcal{H}$, i.e., there is an $r : \mathbb{N} \to S$ s.t.: $\forall x.\, x \in \mathcal{H} \;\leftrightarrow\; \mathcal{F} \vdash rx$
Then $\mathcal{F}$ has an independent sentence rc:

$$\mathcal{F} \nvdash rc \wedge \mathcal{F} \nvdash \neg rc$$

Proof.

$h := d_{\mathcal{F}} \circ r : \mathbb{N} \rightharpoonup \mathbb{B}$ agrees with the halting problem:

$$\forall x.\, d_{\mathcal{F}}\,(rx) \triangleright \textit{tt} \;\leftrightarrow\; \mathcal{F} \vdash rx \;\leftrightarrow\; x \in \mathcal{H},$$

and therefore diverges on some input c. Thus rc is independent in $\mathcal{F}$. $\square$

[10] Kleene 1952.

Going from Soundness to Consistency

- Consider weak representability:

$$\forall x.\, Px \leftrightarrow \mathcal{F} \vdash rx$$

Definition

A formal system $\mathcal{F}'$ is an extension of $\mathcal{F}$, if

$$\forall s.\, \mathcal{F} \vdash s \rightarrow \mathcal{F}' \vdash s$$

- Only transfers along extensions that preserve $\mathcal{F} \vdash rx \rightarrow Px$, i.e., sound extensions
- Can we do better?

Recursively Inseparable Predicates

Theorem

Consider the following predicates:

$$\mathcal{I}_{\mathit{tt}} := \lambda x.\, \theta x x \triangleright \mathit{tt} \qquad \mathcal{I}_{\mathit{ff}} := \lambda x.\, \theta x x \triangleright \mathit{ff}$$

They are recursively inseparable, i.e., any partial function $f : \mathbb{N} \rightharpoonup \mathbb{B}$ s.t.

$$\forall x.\, (x \in \mathcal{I}_{\mathit{tt}} \;\rightarrow\; f x \triangleright \mathit{tt}) \quad \wedge \quad (x \in \mathcal{I}_{\mathit{ff}} \;\rightarrow\; f x \triangleright \mathit{ff})$$

diverges on some input.

Kleene's Improved Incompleteness Proof[11]

Theorem

Assume $\mathcal{F}$ strongly separates $\mathcal{I}_{tt}$ and $\mathcal{I}_{ff}$, i.e., there is an $r : \mathbb{N} \to S$ s.t.:

$$\forall x.\, x \in \mathcal{I}_{tt} \;\to\; \mathcal{F} \vdash rx \quad \wedge \quad x \in \mathcal{I}_{ff} \;\to\; \mathcal{F} \vdash \neg rx$$

Any (consistent) extension $\mathcal{F}'$ of $\mathcal{F}$ has an independent sentence rc:

$$\mathcal{F}' \nvdash rc \wedge \mathcal{F}' \nvdash \neg rc$$

Proof.

$h := d_{\mathcal{F}'} \circ r : \mathbb{N} \rightharpoonup \mathbb{B}$ recursively separates $\mathcal{I}_{tt}$ and $\mathcal{I}_{ff}$, and therefore diverges on some input c. Therefore, rc is independent in $\mathcal{F}'$. □

[11] Kleene 1951, c.f. Kleene 1952; Kleene 1967

Abstract Incompleteness Proofs

Instantiation to first-order Robinson arithmetic

Instantiating the Incompleteness Proofs

From now on: Assume θ in EPF to be an interpreter for μ-recursive functions

Lemma

FA[12] weakly represents any enumerable predicate $P : \mathbb{N} \to \mathbb{P}$ using a Σ_1-formula φ:

$$\forall x.\, Px \;\leftrightarrow\; \mathrm{FA} \vdash \varphi(\overline{x})$$

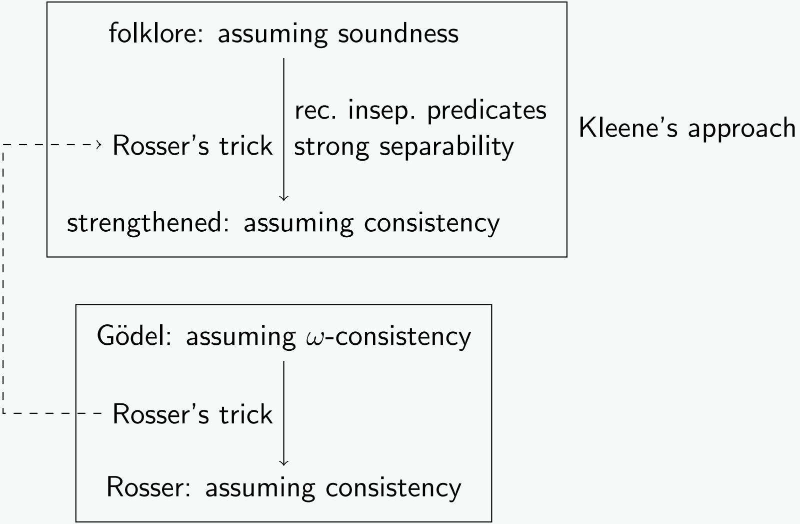
Proof.

See Kirst and Hermes 2021, relying on a mechanisation of the DPRM theorem by Larchey-Wendling and Forster 2022. □

Goal: Show that Robinson arithmetic is strong enough to strongly separate any pair of enumerable and disjoint predicates.

[12] A subset of Robinson arithmetic.

Path Towards Rosser's Trick



Rosser's Trick for Strong Separability

Let P_1, P_2 be enumerable and disjoint predicates, and $\varphi_1, \varphi_2 \in \Delta_0$ such that:

$$P_1\,x \;\leftrightarrow\; \mathrm{Q} \vdash \exists k.\, \varphi_1(\overline{x}, k) \qquad P_2\,x \;\leftrightarrow\; \mathrm{Q} \vdash \exists l.\, \varphi_2(\overline{x}, l)$$

We want to find Φ_1 such that for all x:

$$P_1\,x \;\rightarrow\; \mathrm{Q} \vdash \exists k.\, \Phi_1(\overline{x}, k) \qquad P_2\,x \;\rightarrow\; \mathrm{Q} \vdash \neg\exists k.\, \Phi_1(\overline{x}, k)$$

$$\Phi_1(x, k) := \varphi_1(x, k) \wedge \forall k' \leq k.\, \neg\varphi_2(x, k')$$

Instantiating the Strengthened Incompleteness Proof

Theorem

Robinson arithmetic is essentially incomplete.

$$\forall T \supseteq \mathrm{Q}.\quad T \text{ enumerable} \;\to\; T \nvdash \bot \;\to\; \exists \varphi.\, T \nvdash \varphi \land T \nvdash \neg\varphi$$

$$\forall T \supseteq \mathrm{Q}.\quad T \text{ enumerable} \;\to\; \mathbb{N} \vDash T \;\to\; (\forall \varphi.\, T \vdash \varphi \lor T \vdash \neg\varphi) \;\to\; \mathcal{H}_{\mathrm{TM}} \text{ decidable}$$

Summary

- Gave abstract incompleteness proofs due to Kleene in different strengths, reformulated in synthetic computability
  - Assuming weak representability, using the halting problem
  - Assuming strong separability, using recursively inseparable predicates
  - Mechanised in only about 450 stand-alone lines of Coq, 200 for the strongest result
- Instantiated those proofs to first-order Robinson arithmetic using Rosser's trick
  - Relying on libraries of undecidability[13] and first-order logic[14]
  - Mechanised in around 2200 lines of Coq
- Future Work:
  - Church's thesis for Robinson arithmetic
  - Avoid DPRM
  - Gödel's second incompleteness theorem

[13] Forster et al. 2020.

[14] Kirst, Hostert, et al. 2022.

References I

(https://math.stackexchange.com/users/21820/user21820), user21820 (Dec. 31, 2021). *Computability Viewpoint of Godel/Rosser's Incompleteness Theorem*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/2486349 (visited on 03/22/2022).

Aaronson, Scott (July 21, 2011). *Rosser's theorem via Turing machines*. Shtetl-Optimized. URL: https://scottaaronson.blog/?p=710 (visited on 02/28/2022).

Bauer, Andrej (2006). "First Steps in Synthetic Computability Theory". In: *Electronic Notes in Theoretical Computer Science* 155, pp. 5–31.

Forster, Yannick (2022). "Parametric Church's Thesis: Synthetic Computability Without Choice". In: *International Symposium on Logical Foundations of Computer Science*, pp. 70–89.

References II

- Forster, Yannick et al. (2020). "A Coq Library of Undecidable Problems". In: *CoqPL 2020 The Sixth International Workshop on Coq for Programming Languages.*
- Harrison, John (2009). *Handbook of Practical Logic and Automated Reasoning*. Cambridge University Press.
- Kirst, Dominik and Marc Hermes (2021). "Synthetic Undecidability and Incompleteness of First-Order Axiom Systems in Coq". In: *ITP 2021.*
- Kirst, Dominik, Johannes Hostert, et al. (2022). "A Coq Library for Mechanised First-Order Logic". In: *The Coq Workshop.*
- Kleene, Stephen C. (1936). "General Recursive Functions of Natural Numbers". In: *Mathematische Annalen* 112, pp. 727–742.
- — (1951). "A Symmetric Form of Gödel's theorem". In: *The Journal of Symbolic Logic* 16.2, p. 147.
- — (1952). *Introduction to Metamathematics*. North Holland.

References III

- Kleene, Stephen C. (1967). *Mathematical Logic*. Dover Publications.
- Kreisel, Georg (1967). “Mathematical Logic”. In: *Journal of Symbolic Logic* 32.3, pp. 419–420.
- Larchey-Wendling, Dominique and Yannick Forster (2022). “Hilbert’s Tenth Problem in Coq (Extended Version)”. In: *Logical Methods in Computer Science* 18.
- O’Connor, Russell (2005). “Essential Incompleteness of Arithmetic Verified by Coq”. In: *Theorem Proving in Higher Order Logics*, pp. 245–260.
- Paulson, Lawrence C. (2014). “A Machine-Assisted Proof of Gödel’s Incompleteness Theorems for the Theory of Hereditarily Finite Sets”. In: *The Review of Symbolic Logic* 7.3, pp. 484–498.
- Popescu, Andrei and Dmitriy Traytel (2019). “A Formally Verified Abstract Account of Gödel’s Incompleteness Theorems”. In: *Automated Deduction – CADE 27*. Springer International Publishing, pp. 442–461.

References IV

- Richman, Fred (1983). "Church's Thesis Without Tears". In: *The Journal of Symbolic Logic* 48.3, pp. 797–803.
- Shankar, Natarajan (1994). *Metamathematics, Machines and Gödel's Proof*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press.
- Troelstra, Anne S. and Dirk van Dalen (1988). *Constructivism in Mathematics, Vol 1*. ISSN. Elsevier Science.
- Turing, Alan M. (1936). "On Computable Numbers, with an Application to the Entscheidungsproblem". In: *Proceedings of the London Mathematical Society* 2.42, pp. 230–265.

Church's thesis

$$\forall f : \mathbb{N} \rightharpoonup \mathbb{N}.\, \exists \varphi \in \Sigma_1.\, \forall x y.\, f x \triangleright y \;\leftrightarrow\; Q \vdash \forall y'.\, \varphi(\overline{x}, y') \;\leftrightarrow\; y = y'$$