Gödel's Theorem Without Tears

Essential Incompleteness in Synthetic Computability

Final Bachelor Talk

Benjamin Peters

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17th June, 2022

Gödel's Theorem Without Tears[1]

Essential Incompleteness in Synthetic Computability

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Benjamin Peters Dominik Kirst

SAARLAND UNIVERSITY
COMPUTER SCIENCE

[1] Abstract title: "Strong, Synthetic, and Computational Proofs of Gödel's First Incompleteness Theorem"

Gödel's First Incompleteness Theorem

Gödel's first incompleteness theorem[2]

Any effective, sound, and sufficiently powerful formal logic is incomplete.

[2] Gödel 1931.

Gödel's First Incompleteness Theorem

Gödel-Rosser incompleteness theorem[2]

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- We instantiate these results to first-order Robinson arithmetic
- All results have been mechanised in Coq[4]

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[3] Shankar 1994; O'Connor 2005; Harrison 2009; Paulson 2014; Popescu and Traytel 2019.
[4] `https://github.com/uds-psl/coq-synthetic-incompleteness/tree/bachelor`

Abstract Incompleteness Proofs

Instantiation to first-order Robinson arithmetic

Synthetic Computability[5]

We work in CIC, where we can consider the function space to only contain computable functions

[5] Richman 1983; Bauer 2006.

Synthetic Computability[5]

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Definition

A predicate $P : X \to \mathbb{P}$ is

- enumerable if $\exists f : \mathbb{N} \to \mathcal{O}(X).\, Px \;\leftrightarrow\; \exists k.\, fk = \ulcorner x \urcorner$.

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- decidable if $\exists f : X \to \mathbb{B}.\, Px \;\leftrightarrow\; fx = \mathit{tt}$.

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Formal Systems

Definition (Formal system)

$\mathcal{F} = (S, \neg, \vdash)$ is a formal system if:

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First-order logic over a consistent and enumerable axiomatisation is a formal system in this sense

Decidable Formal Systems

Lemma

There is a partial function $d_{\mathcal{F}} : S \rightharpoonup \mathbb{B}$ separating provability from refutability:

$$\forall s. \left(d_{\mathcal{F}}\, s \triangleright \mathit{tt} \;\leftrightarrow\; \mathcal{F} \vdash s\right) \land \left(d_{\mathcal{F}}\, s \triangleright \mathit{ff} \;\leftrightarrow\; \mathcal{F} \vdash \neg s\right)$$

If $\mathcal{F}$ is complete, $d_{\mathcal{F}}$ is total.

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If $\mathcal{F}$ is complete, $d_{\mathcal{F}}$ is total.

Corollary

Any complete formal system is decidable.

Kleene's Folklore Incompleteness Proof[6,7]

Theorem

Let $\mathcal{F}$ be complete and weakly represent $P : \mathbb{N} \to \mathbb{P}$, i.e., there is an $r : \mathbb{N} \to S$ s.t.:

$$\forall x. \, Px \;\leftrightarrow\; \mathcal{F} \vdash rx$$

Then P is decidable.

[6] Kleene 1936; Turing 1936.

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Then P is decidable. Thus, if P is undecidable, $\mathcal{F}$ is incomplete.

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Church's Thesis[9]

Axiom (EPF[8])

There is a function $\theta : \mathbb{N} \to \mathbb{N} \rightharpoonup \mathbb{B}$ such that:

$$\forall f : \mathbb{N} \rightharpoonup \mathbb{B}.\, \exists c.\, f \equiv \theta c$$

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Definition (Self-halting problem)

The self-halting problem is defined as:

$$\mathcal{H} := \lambda x.\, \exists b.\, \theta x x \triangleright b$$

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Self-halting problem

Fact

Partial functions $f : \mathbb{N} \rightharpoonup \mathbb{B}$ agreeing with the halting problem $\mathcal{H} := \lambda x.\, \exists b.\, \theta x x \triangleright b$:

$$\forall x.\, x \in \mathcal{H} \;\leftrightarrow\; f x \triangleright \boldsymbol{tt},$$

diverge on some input c, i.e., $\forall b.\, f c \not\triangleright b$.

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Proof.

Consider $g : \mathbb{N} \rightharpoonup \mathbb{B}$,

$$g x := \begin{cases} \mathit{ff} & \text{if } f x \triangleright \mathit{tt} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Let c be the code of g. We have $f c \triangleright \mathit{tt} \;\leftrightarrow\; f c \triangleright \mathit{ff}$. □

Strengthening the Folklore Proof[10]

Theorem

Assume $\mathcal{F}$ weakly represents $\mathcal{H}$, i.e., there is an $r : \mathbb{N} \to S$ s.t.: $\forall x.\, x \in \mathcal{H} \;\leftrightarrow\; \mathcal{F} \vdash rx$
Then $\mathcal{F}$ has an independent sentence rc:

$$\mathcal{F} \nvdash rc \wedge \mathcal{F} \nvdash \neg rc$$

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Proof.

$h := d_{\mathcal{F}} \circ r : \mathbb{N} \rightharpoonup \mathbb{B}$ agrees with the halting problem:

$$\forall x.\, d_{\mathcal{F}}\,(rx) \triangleright \textit{tt} \;\leftrightarrow\; \mathcal{F} \vdash rx \;\leftrightarrow\; x \in \mathcal{H},$$

and therefore diverges on some input c. Thus rc is independent in $\mathcal{F}$. □

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Going from Soundness to Consistency

- Consider weak representability:

$$\forall x.\, Px \;\leftrightarrow\; \mathcal{F} \vdash rx$$

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- Only transfers along extensions that preserve $\mathcal{F} \vdash rx \;\rightarrow\; Px$, i.e., sound extensions

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- Can we do better?

Recursively Inseparable Predicates

Theorem

Consider the following predicates:

$$\mathcal{I}_{tt} := \lambda x.\, \theta x x \triangleright tt \qquad \mathcal{I}_{ff} := \lambda x.\, \theta x x \triangleright ff$$

They are recursively inseparable, i.e., any partial function $f : \mathbb{N} \rightharpoonup \mathbb{B}$ s.t.

$$\forall x.\, (x \in \mathcal{I}_{tt} \ \rightarrow \ f x \triangleright tt) \quad \wedge \quad (x \in \mathcal{I}_{ff} \ \rightarrow \ f x \triangleright ff)$$

diverges on some input.

Kleene's Improved Incompleteness Proof [11]

Theorem

Assume $\mathcal{F}$ strongly separates $\mathcal{I}_{tt}$ and $\mathcal{I}_{ff}$, i.e., there is an $r : \mathbb{N} \to S$ s.t.:

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Proof.

$h := d_{\mathcal{F}} \circ r : \mathbb{N} \rightharpoonup \mathbb{B}$ recursively separates $\mathcal{I}_{tt}$ and $\mathcal{I}_{ff}$, and therefore diverges on some input c. Therefore, rc is independent in $\mathcal{F}$. □

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Any (consistent) extension $\mathcal{F}'$ of $\mathcal{F}$ has an independent sentence rc:

$$\mathcal{F}' \nvdash rc \wedge \mathcal{F}' \nvdash \neg rc$$

Proof.

$h := d_{\mathcal{F}'} \circ r : \mathbb{N} \rightharpoonup \mathbb{B}$ recursively separates $\mathcal{I}_{tt}$ and $\mathcal{I}_{ff}$, and therefore diverges on some input c. Therefore, rc is independent in $\mathcal{F}'$. □

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Instantiation to first-order Robinson arithmetic

Instantiating the Incompleteness Proofs

From now on: Assume θ in EPF to be an interpreter for μ-recursive functions

[12]A subset of Robinson arithmetic.

Instantiating the Incompleteness Proofs

From now on: Assume θ in EPF to be an interpreter for μ-recursive functions

Lemma

FA[12] weakly represents any enumerable predicate $P : \mathbb{N} \to \mathbb{P}$ using a Σ_1-formula φ:

$$\forall x.\, Px \;\leftrightarrow\; \mathrm{FA} \vdash \varphi(\overline{x})$$

Proof.

See Kirst and Hermes 2021, relying on a mechanisation of the DPRM theorem by Larchey-Wendling and Forster 2022. □

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Goal: Show that Robinson arithmetic is strong enough to strongly separate any pair of enumerable and disjoint predicates.

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Path Towards Rosser's Trick

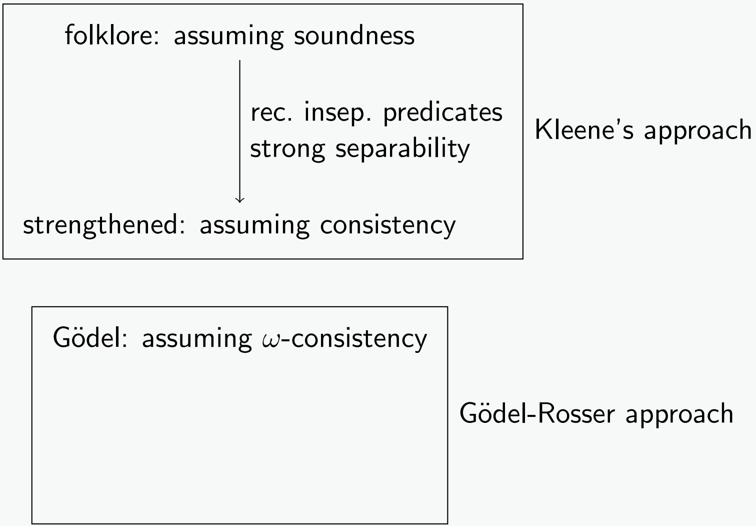
folklore: assuming soundness

↓ rec. insep. predicates
strong separability

strengthened: assuming consistency

Kleene's approach

Path Towards Rosser's Trick



Path Towards Rosser's Trick

folklore: assuming soundness

↓ rec. insep. predicates, strong separability

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Kleene's approach

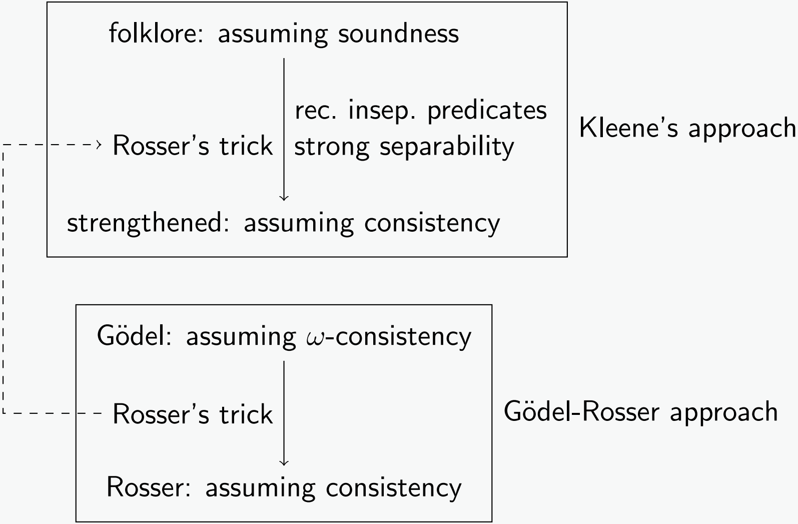
Gödel: assuming ω-consistency

↓ Rosser's trick

Rosser: assuming consistency

Gödel-Rosser approach

Path Towards Rosser's Trick



Rosser's Trick for Strong Separability

Let P_1, P_2 be enumerable and disjoint predicates, and $\varphi_1, \varphi_2 \in \Delta_0$ such that:

$$P_1\,x \;\leftrightarrow\; \mathrm{Q} \vdash \exists k.\, \varphi_1(\overline{x}, k) \qquad P_2\,x \;\leftrightarrow\; \mathrm{Q} \vdash \exists l.\, \varphi_2(\overline{x}, l)$$

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We want to find Φ_1 such that for all x:

$$P_1\,x \;\rightarrow\; \mathrm{Q} \vdash \exists k.\, \Phi_1(\overline{x}, k) \qquad P_2\,x \;\rightarrow\; \mathrm{Q} \vdash \neg\exists k.\, \Phi_1(\overline{x}, k)$$

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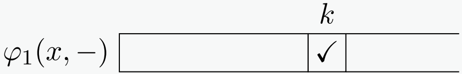

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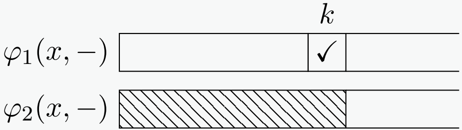

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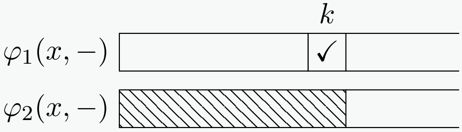

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Instantiating the Strengthened Incompleteness Proof

Theorem

Robinson arithmetic is essentially incomplete.

$$\forall T \supseteq \mathrm{Q}.\quad T \text{ enumerable} \;\rightarrow\; T \nvdash \bot \;\rightarrow\; \exists \varphi.\, T \nvdash \varphi \land T \nvdash \neg\varphi$$

Instantiating the Strengthened Incompleteness Proof

Theorem

Robinson arithmetic is essentially incomplete.

$$\forall T \supseteq \mathrm{Q}.\quad T \text{ enumerable} \to \quad T \nvdash \bot \to \qquad \exists \varphi.\, T \nvdash \varphi \land T \nvdash \neg\varphi$$

$$\forall T \supseteq \mathrm{Q}.\quad T \text{ enumerable} \to \quad \mathbb{N} \vDash T \to \quad (\forall \varphi.\, T \vdash \varphi \lor T \vdash \neg\varphi) \to \mathcal{H}_{\mathrm{TM}} \text{ decidable}$$

Summary

- Gave abstract incompleteness proofs due to Kleene in different strengths, reformulated in synthetic computability
  - Assuming weak representability, using the halting problem
  - Assuming strong separability, using recursively inseparable predicates
  - Mechanised in only about 450 stand-alone lines of Coq, 200 for the strongest result

[13] Forster et al. 2020.
[14] Kirst, Hostert, et al. 2022.

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- Future Work:
  - Church's thesis for Robinson arithmetic
  - Avoid DPRM
  - Gödel's second incompleteness theorem

[13] Forster et al. 2020.

[14] Kirst, Hostert, et al. 2022.

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Church's thesis

$$\forall f : \mathbb{N} \rightharpoonup \mathbb{N}.\, \exists \varphi \in \Sigma_1.\, \forall xy.\, f x \triangleright y \;\leftrightarrow\; Q \vdash \forall y'.\, \varphi(\overline{x}, y') \;\leftrightarrow\; y = y'$$