Strong, Synthetic, and Computational Proofs of Gödel's First Incompleteness Theorem

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There is a simple folklore proof of Gödel's first incompleteness theorem (G1) by Kleene using computability theory and undecidability of the halting problem [8]. As opposed to Gödel's original proof [6], which directly arithmetizes provability, and Rosser's improvement on this result by modifying the provability predicate [17], Kleene's proof is much easier to spell out in detail, only relying on basic results in computability theory [9].

Constructive logics are a useful tool for formalizing results in computability theory because in such logics usually all definable functions are computable, avoiding the need to argue via a concrete model of computation. They therefore appear to provide an elegant way to formalize G1 when combined with Kleene's folklore proof [7]. However, Kleene's proof only shows a considerably weaker statement: It only works for sound as opposed to just consistent formal systems, and does not construct an independent sentence.

Nevertheless, Kleene did find a way to fix these weaknesses using Rosser's trick [10, 9]. However, this result is much less well-known, as evidenced by [1, 20].

We first outline how to formalize both the folklore and the strengthened versions of Kleene's incompleteness proofs for abstract formal systems in the calculus of inductive constructions (CIC) [3, 14]. To do this synthetically, we assume the axiom of Church's thesis [12, 19, 4], internalizing the fact that all constructively definable functions are computable. Secondly, we instantiate these proofs with a concrete presentation of first-order logic using Rosser's trick.

Most of the results presented here have been mechanized using the Coq proof assistant [18].

Synthetic computability. We are using *synthetic computability theory* [15, 2, 5] to formalize our results without directly working with a concrete model of computation. We write $X^?$ for the option type $X + \mathbb{1}$, containing values $^\circ x$ and a none value. We say that a predicate $P : X \to \mathbb{P}$ is enumerable if there is a function $f : \mathbb{N} \to X^?$ such that $\forall x.\, Px \;\leftrightarrow\; \exists k.\, fk = {^\circ x}$, and decidable if there is a function $f : X \to \mathbb{B}$ such that $\forall x.\, Px \;\leftrightarrow\; fx = tt$. We also work with a type of partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$. It can for example be realized using step indexed functions $\mathbb{N} \to \mathbb{N} \to \mathbb{N}^?$. We write $fx \triangleright y$ if fx halts and evaluates to y.

Weak G1. Our abstract notion of a formal system consists of an enumerable and discrete type of sentences S, an enumerable provability predicate $\vdash\, : S \to \mathbb{P}$, and a negation function $\neg : S \to S$ such that $\vdash$ is consistent: $\forall s.\, \neg(\vdash s \wedge \vdash \neg s)$. We call a formal system complete if $\forall s.\vdash s \vee \vdash \neg s$. Note that in a complete formal system, provability is decidable.

We say that a formal system weakly represents a predicate $P : \mathbb{N} \to \mathbb{P}$ if there is a representation function $R_P : \mathbb{N} \to S$ such that $\forall x.\, Px \;\leftrightarrow\; \vdash R_P\, x$.

Assume that there is a formal system that is complete and weakly represents the halting problem H for some model of computation. Now, H is decidable, because $\lambda x.\vdash R_H\, x$ is decidable, since the formal system is complete. This is the folklore proof of G1, as mechanized in [7].

There are multiple ways in which we strengthen this result, following Kleene:

- Instead of decidability of the halting problem, we derive falsity from completeness.

- There are unsound (but consistent) formal systems that do not weakly represent H because the direction from right to left requires a form of soundness. We show incompleteness even for such formal systems by modifying the representability property required.
- We explicitly construct an independent sentence, that is, we show $\exists s. \nvdash s \land \nvdash \neg s$.

Unfortunately, the above notions from synthetic computability theory are not strong enough to obtain these results without directly working with a concrete model of computation.

Church's thesis. We can however internalize the notion that all definable functions are computable by assuming a formulation of the axiom of Church's thesis (CT) [12, 19, 4], that is, in our case, a universal function $\theta : \mathbb{N} \to (\mathbb{N} \rightharpoonup \mathbb{N})$ such that:

$$\forall f : \mathbb{N} \rightharpoonup \mathbb{N}. \exists c. \forall xy. \theta cx \triangleright y \ \leftrightarrow \ fx \triangleright y$$

We can now show that the halting problem $H := \lambda c. \exists y. \theta cc \triangleright y$ is undecidable. We can either assume θ to be abstract or to be an interpreter of a Turing-complete model of computation [11].

Strong G1. For Kleene's stronger result, consider the following two predicates:

$$A_1 := \{x \mid \theta xx \triangleright 1\} \qquad\qquad A_0 := \{x \mid \theta xx \triangleright 0\}$$

A_1 and A_0 are enumerable and recursively inseparable, that is, there is no decidable predicate D such that $A_1 \subseteq D$ and $A_0 \subseteq \overline{D}$. We assume that the formal system strongly separates A_1 and A_0, that is, there is a representation function $R : \mathbb{N} \to S$ such that:

$$x \in A_1 \ \to \ \vdash Rx \qquad\qquad x \in A_0 \ \to \ \vdash \neg Rx$$

Note that we do not need any form of soundness anymore. Consider a partial function that checks whether $\vdash Rx$ or $\vdash \neg Rx$ by enumerating all provable sentences and outputting 1 or 0 respectively. This function must diverge on some input c, because it would separate A_1 and A_0 otherwise, and therefore $\nvdash Rc$ and $\nvdash \neg Rc$. This input can be constructed explicitly using diagonalization and an application of CT. This would not be possible had we not assumed CT.

Instantiation. We use the same framework for first-order logic as in the instantiation of the folklore proof [7] with the theory of Robinson's Q [16]. We instantiate θ with an interpreter for μ-recursive functions, as described in [13]. Q weakly represents all predicates enumerable in μ (and by CT, all synthetically enumerable predicates) using Σ_1 formulas [13, 7].

Let φ_1, φ_0 be Σ_1-formulas that weakly represent A_1, A_0 respectively, that is $\forall c. c \in A_i \ \leftrightarrow Q \vdash \varphi_i(c)$. We can concretely assume that $\varphi_i(x) = \exists k. \psi_i(x,k)$, where ψ_i is Q-decidable, that is $\mathsf{Q} \vdash \psi_i(x,k) \lor \mathsf{Q} \vdash \neg\psi_i(x,k)$. We now apply Rosser's trick to φ_i, that is, we choose:

$$\Phi_i(x) := \exists k. \psi_i(x,k) \land \forall k' \leq k. \neg\psi_{1-i}(x,k')$$

Intuitively, Φ_i can be understood as "There is a proof k of $x \in A_i$, and there is no smaller proof of $x \in A_{1-i}$". Now, Φ_1 and Φ_0 both strongly separate A_1 and A_0:

$$x \in A_i \ \to \ \mathsf{Q} \vdash \Phi_i(x) \qquad\qquad x \in A_{1-i} \ \to \ \mathsf{Q} \vdash \neg\Phi_i(x)$$

Just as Rosser's trick allowed weakening the precondition of ω-consistency in Gödel's original proof of G1, it allows us to drop the requirement of soundness for the theory we are working with by relying on another form of representability. All properties required also hold for consistent extensions of Q, which allows us to show essential incompleteness of Q. It is also possible to obtain essential undecidability of Q by modifying the abstract results slightly.

This approach can be used to show incompleteness of other formal systems, such as CIC or other higher-order logics, as long as they weakly represent H and can apply Rosser's trick.

References

[1] Scott Aaronson. *Rosser's theorem via Turing machines*. Shtetl-Optimized. URL: https://scottaaronson.blog/?p=710 (visited on 28/02/2022).

[2] Andrej Bauer. "First Steps in Synthetic Computability Theory". In: *Electronic Notes in Theoretical Computer Science* 155 (2006), pp. 5–31.

[3] Thierry Coquand and Gérard Huet. "The calculus of constructions". In: *Information and Computation* 76.2 (1988), pp. 95–120.

[4] Yannick Forster. "Parametric Church's Thesis: Synthetic Computability Without Choice". In: *International Symposium on Logical Foundations of Computer Science*. 2022, pp. 70–89.

[5] Yannick Forster, Dominik Kirst and Gert Smolka. "On Synthetic Undecidability in Coq, with an Application to the Entscheidungsproblem". In: *Proceedings of the 8th ACM SIGPLAN International Conference on Certified Programs and Proofs*. 2019, pp. 38–51.

[6] Kurt Gödel. "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I". In: *Monatshefte für Mathematik und Physik* 38 (1931), pp. 173–198.

[7] Dominik Kirst and Marc Hermes. "Synthetic Undecidability and Incompleteness of First-Order Axiom Systems in Coq". In: *ITP 2021*. 2021.

[8] Stephen C. Kleene. "General recursive functions of natural numbers." In: *Mathematische Annalen* 112 (1936), pp. 727–742.

[9] Stephen C. Kleene. *Mathematical Logic*. Dover Publications, 1967.

[10] Stephen C. Kleene. "Recursive predicates and quantifiers". In: *Transactions of the American Mathematical Society* 53 (1943), pp. 41–73.

[11] Georg Kreisel. "Church's Thesis: A Kind of Reducibility Axiom for Constructive Mathematics". In: *Studies in Logic and the Foundations of Mathematics* 60 (1970), pp. 121–150.

[12] Georg Kreisel. "Mathematical Logic". In: *Journal of Symbolic Logic* 32.3 (1967), pp. 419–420.

[13] Dominique Larchey-Wendling and Yannick Forster. "Hilbert's Tenth Problem in Coq (Extended Version)". In: *Logical Methods in Computer Science* 18 (2022).

[14] Christine Paulin-Mohring. "Inductive definitions in the system Coq rules and properties". In: *Typed Lambda Calculi and Applications*. Springer, 1993, pp. 328–345.

[15] Fred Richman. "Church's Thesis Without Tears". In: *The Journal of Symbolic Logic* 48.3 (1983), pp. 797–803.

[16] Raphael Robinson. "An Essentially Undecidable Axiom System". In: *Proceedings of the International Congress of Mathematics*. 1950, pp. 729–730.

[17] Barkley Rosser. "Extensions of some theorems of Gödel and Church". In: *Journal of Symbolic Logic* 1.3 (1936), pp. 87–91.

[18] The Coq Development Team. *The Coq Proof Assistant*. Jan. 2022. DOI: 10.5281/zenodo.5846982.

[19] Anne S. Troelstra and Dirk van Dalen. *Constructivism in Mathematics, Vol 1*. ISSN. Elsevier Science, 1988.

[20] Anatoly Vorobey. *First Incompleteness via computation: an explicit construction*. Foundations of Mathematics mailing list. URL: https://cs.nyu.edu/pipermail/fom/2021-September/022872.html (visited on 21/02/2022).