Terminating Tableaux for Modal Logic with Transitive Closure

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Bachelor's Thesis Final Talk
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Modal Logic with Transitive Closure: K*

$$\begin{aligned} t &::= p \mid \dot{\neg} t \mid t \mathbin{\dot{\wedge}} t \mid t \mathbin{\dot{\vee}} t \mid \Diamond \rho t \mid \Box \rho t \\ \rho &::= r \mid r^* \end{aligned}$$

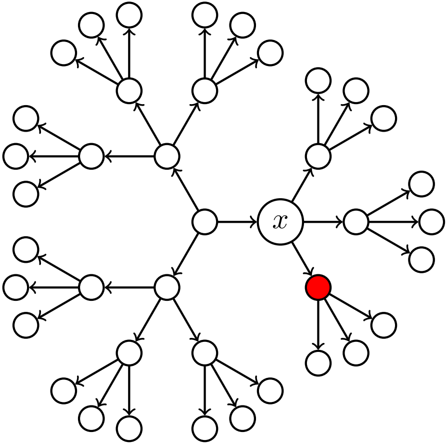
- Extends basic modal logic K with reflexive transitive closure operator
- Fragment of Propositional Dynamic Logic

K* as Fragment of PDL

- Model computation as state transition
- Programs are transitions, programs are represented in the logic
- Provide operators to compose new programs

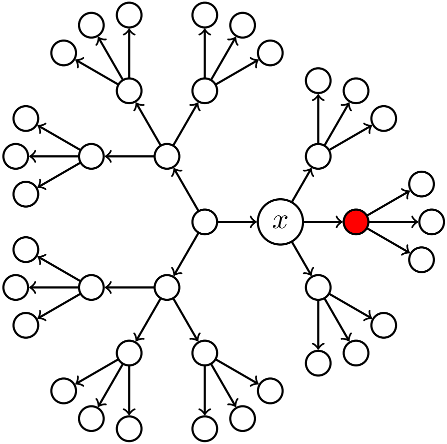
$; : (\iota\iota o)(\iota\iota o)\iota\iota o$	Sequentialization
$\cap : (\iota\iota o)(\iota\iota o)\iota\iota o$	Choice
$^* : (\iota\iota o)\iota\iota o$	Iteration
$? : (\iota o)\iota\iota o$	Test

Semantics



$\Diamond rpx$

Semantics



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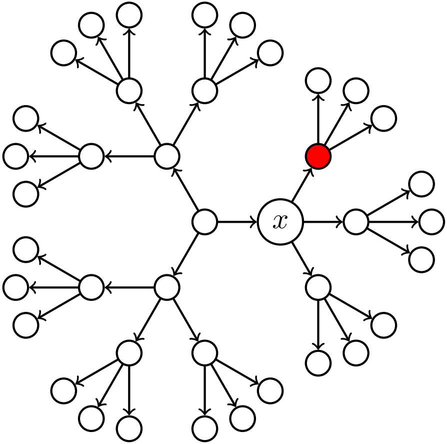

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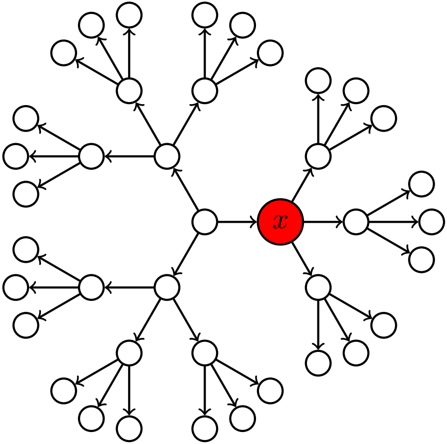
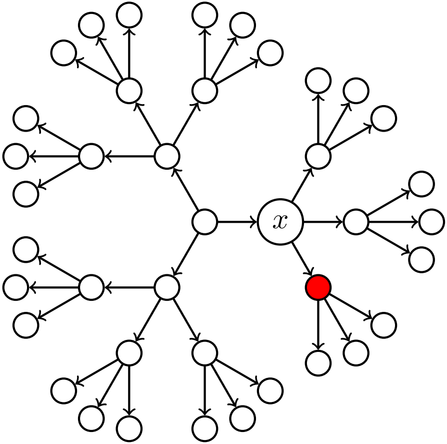


$\Diamond r^* p x$

Semantics



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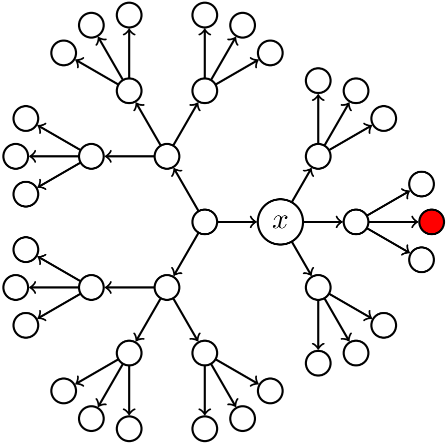

$\Diamond r^* p x$

- Small Model Property
- Decidable
- Not Compact: $\{\Diamond r^* p, \dot{\neg} p, \Box r \dot{\neg} p, \Box r (\Box r \dot{\neg} p), \dots \}$

Split the propositional variables into two disjoint sets

- Path variables denoted by $\mathcal{V}$
- Proper propositional variables

Definition (Extended Grammar)

$$\begin{array}{rcll} s & ::= & ux \mid \Diamond r\alpha x \mid \alpha = \Diamond r^* t \mid rxx & \textbf{formula} \\ u & ::= & \alpha \mid t & \textbf{extended modal expression} \\ t & ::= & p \mid \dot{\neg} t \mid t \mathbin{\dot{\wedge}} t \mid t \mathbin{\dot{\vee}} t \mid \Diamond \rho t \mid \Box \rho t & \textbf{proper modal expression} \\ \rho & ::= & r \mid r^* & \end{array}$$

A tableau system

$$\mathcal{T}_{\dot\neg}\ \frac{(\dot\neg p)x,\ px}{\quad} \qquad \mathcal{T}_{\dot\wedge}\ \frac{(t_1 \mathbin{\dot\wedge} t_2)x}{t_1x\ ,\ t_2x} \qquad \mathcal{T}_{\dot\vee}\ \frac{(t_1 \mathbin{\dot\vee} t_2)x}{t_1x \mid t_2x}$$

$$\mathcal{T}_{\Box}\ \frac{\Box rtx, rxy}{ty} \qquad \mathcal{T}^{R}_{\Box *}\ \frac{\Box r^*tx}{tx} \qquad \mathcal{T}^{T}_{\Box *}\ \frac{\Box r^*tx, rxy}{\Box r^*ty}$$

$$\mathcal{T}_{\Diamond}\ \frac{\Diamond rux}{rxy\ ,\ uy}\ \ y \notin \mathcal{N}_{\iota}A$$

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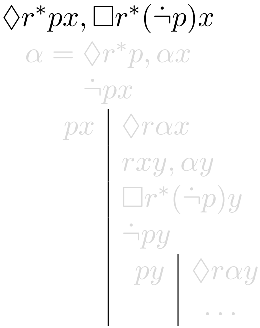
A infinite derivation for an unsatisfiable formula



$$\Diamond r^* px, \Box r^* (\dot\neg p)x$$

$$\alpha = \Diamond r^* p, \alpha x$$

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$$px \;\Big|\; \Diamond r \alpha x$$

$$rxy, \alpha y$$

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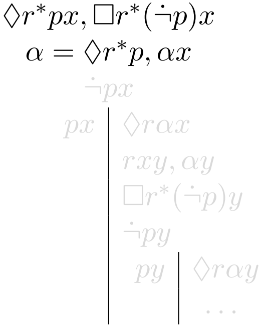

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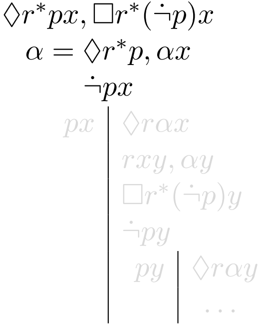

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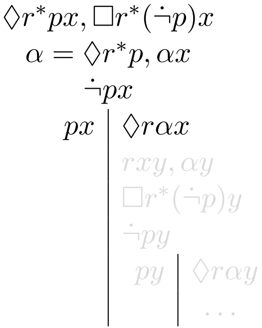

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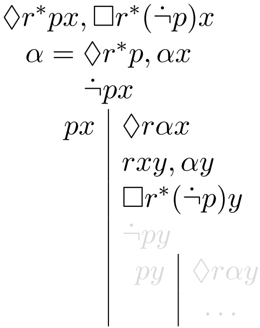

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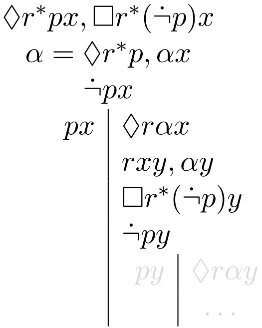

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Problems

- The System does not terminate.
- The System is not complete.
- We need a **soundness** argument to discard the rightmost branch.

Overview

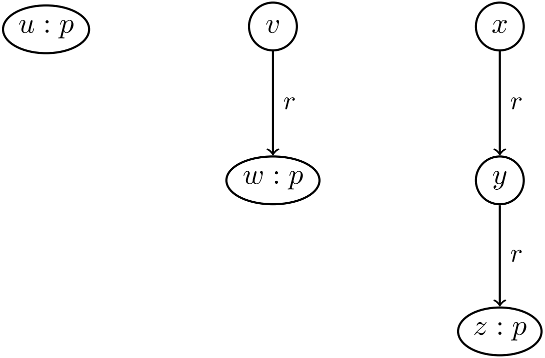
The soundness argument for discarded branches is **straightness**: Preservation of straight branches.

Proof Sketch

- For every satisfiable set of K^*-expressions the initial branch is a straight branch.
- If the premise of a rule is a straight branch, at least one of the rules' alternatives is a straight branch.
- Model existence theorem for straight, maximal branches (w.r.t. applied blocking technique).

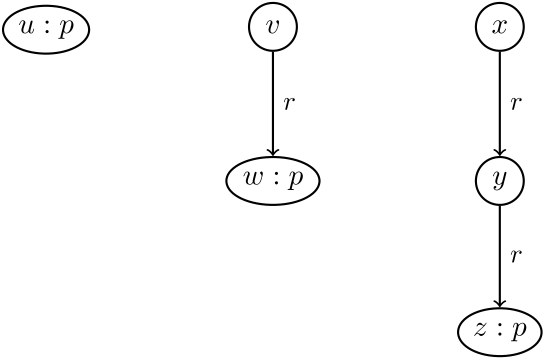
This essentially amounts to deciding existence of a straight, maximal branch instead of satisfiability.

Witness Distance



$\Diamond r^* p$

Witness Distance



$\Diamond r^* p$

$$\delta^r_{\mathcal{I},t} a := \min\{n \in \mathbb{N} \mid \exists b \in \mathcal{I}\iota : a \xrightarrow{r}{}^{n}_{\mathcal{I}} b \wedge \hat{\mathcal{I}} t b = 1\}$$

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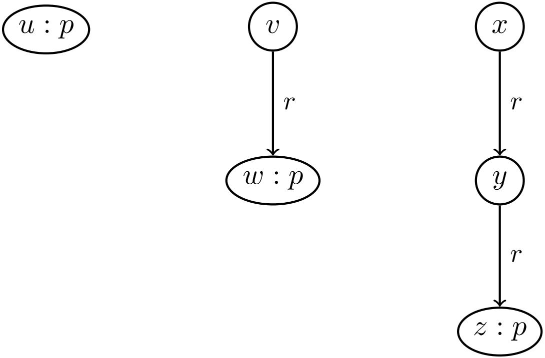

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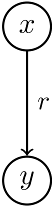
$$\Delta^r_{\mathcal{I},t} L := \min\{\delta^r_{\mathcal{I},t} a \mid \mathcal{I}, a \models L\}$$

Requests



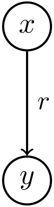
$\Box r(\dot{\neg} p), \Box r^* q$

Definition

Let A be a branch and x be a nominal.

$$\mathcal{R}^r_A x := \{t \mid \Box r t x \in A\} \cup \{t, \Box r^* t \mid \Box r^* t x \in A\}$$

Requests



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$\dot{\neg} p, q, \Box r^* q$

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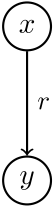

$\Box r(\dot\neg p), \Box r^* q$

$\dot\neg p, q, \Box r^* q$

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Intuition

Use an interpretation to guide the tableau derivation: For each nominal, find a corresponding state in the interpretation to guide branching decisions.

Obey the following rules

O2 Only expand αx to $\Diamond r\alpha x$, if $\mathcal{I}$ does not satisfy the witness at x.

O1 If $\Diamond r\alpha x$ is expanded, then model the successor after a state with optimal witness distance for the witness in $\mathcal{I}$.

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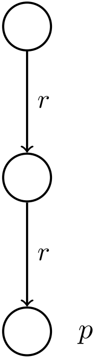
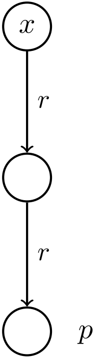
Example

$\Diamond r^* p \quad , \dot{\neg} p$



Example

$\Diamond r^* p x, \dot{\neg} p x$



Example

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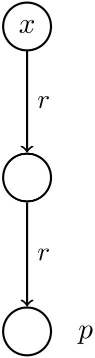

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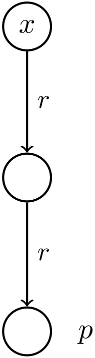

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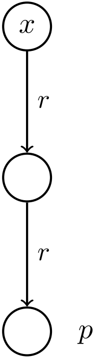

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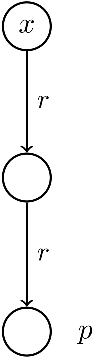
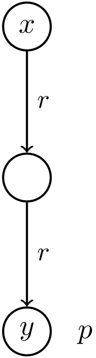
$$\Diamond r \alpha x$$

$$r x y, \alpha y$$



Example

$$\Diamond r^* p x, \dot{\neg} p x$$
$$\alpha = \Diamond r^* p, \alpha x$$
$$\Diamond r \alpha x$$
$$r x y, \alpha y$$



Example

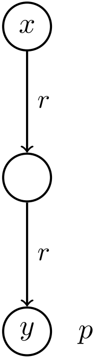
$$\Diamond r^* px, \dot{\neg} px$$

$$\alpha = \Diamond r^* p, \alpha x$$

$$\Diamond r \alpha x$$

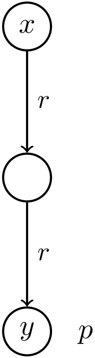
$$rxy, \alpha y$$

$$py \;\Big|\; \Diamond r \alpha y$$



Example

$$\Diamond r^* p x, \dot{\neg} p x$$
$$\alpha = \Diamond r^* p, \alpha x$$
$$\Diamond r \alpha x$$
$$r x y, \alpha y$$
$$p y$$



Straight Branches

Definition

Let A be a branch and $\mathcal{I}$ be an interpretation. $\mathcal{I}$ is **straight** for A if it satisfies the following conditions:

S1 $s \in A \implies \mathcal{I} \models s$ if s is no transition

S2 $rxy \in A \implies \mathcal{I}, \mathcal{I}y \models \mathcal{R}^r x$

O1 $\alpha x, rxy, \alpha y, \alpha = \Diamond r^* t \in A \implies \delta^r_{\mathcal{I},t}(\mathcal{I}y) = \Delta^r_{\mathcal{I},t}(\mathcal{R}^r x)$

O2 $\alpha x, \alpha = \Diamond r^* t, \Diamond r \alpha x \in A \implies \mathcal{I} \not\models tx$

We say A is **straight** if there is an interpretation that is straight for A.

On straight branches, all decisions have been made as if the derivation was guided by $\mathcal{I}$.

Invariant

Straightness

For every rule, if the premise is a straight branch,
at least one of the conclusions is a straight branch.

This is soundness with respect to straight branches.

Straightness Theorem

$$x \xrightarrow{\Diamond r^* t}_A y \iff \exists \alpha \in \mathcal{V}A\colon \alpha = \Diamond r^* t, \alpha x, \alpha y \in A \land x \xrightarrow{r}_A y$$

Theorem 1

Let A be an admissible, straight branch, and $\mathcal{I}$ be straight for A. If $x \xrightarrow{\Diamond r^* t}_A y$ and $\mathcal{I} \not\models ty$, then $\Delta^r_{\mathcal{I},t}(\mathcal{R}^r_A x) > \Delta^r_{\mathcal{I},t}(\mathcal{R}^r_A y)$.

If we could not place the witness, then at least we made progress.

Prove Sketch

- Define a **request relation** w.r.t. blocking technique.
- Prove: If a formula $\Diamond r^* t x \in A$ is not evident, then there is a cycle in a request relation.
- Prove using Theorem 1: If A is straight, then no request relation in A is cyclic.

Approach scales to both pattern- and chain-based blocking. For pattern based blocking the request relation gets more complicated.

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Request Relation: Example

Patterns

Definition (Pattern of a $\Diamond$-Formula)

Let A be a branch. The **$\Diamond$-pattern of a formula** $\Diamond rux \in A$ denoted by $\mathcal{P}^r_A(\Diamond rux)$ is defined according to the following equations:

$$\begin{aligned}\mathcal{P}^r_A(\Diamond rtx) &:= (\{\Diamond rt\}, \mathcal{R}^r_A x)\\ \mathcal{P}^r_A(\Diamond r\alpha x) &:= (\{\Diamond r^* t \mid \alpha = \Diamond r^* t \in A\}, \mathcal{R}^r_A x)\end{aligned}$$

Admissibility Conditions ensure that $\{\Diamond r^* t \mid \alpha = \Diamond r^* t \in A\}$ is always a singleton set.

Realization

Definition ($\Diamond$-Pattern Realization)

Let A be a branch and $x \in \mathcal{N}_\iota A$.

- $(\{\Diamond r t\}, \mathcal{R}^r_A x)$ is **realized** in A, if there is $x', y \in \mathcal{N}_\iota A$ such that $\mathcal{R}^r_A x \subseteq \mathcal{R}^r_A x'$ and $x' \xrightarrow{r}_A y$.
- $(\{\Diamond r^* t\}, \mathcal{R}^r_A x)$ is **realized** in A, if there is $x', y \in \mathcal{N}_\iota A$ such that $\mathcal{R}^r_A x = \mathcal{R}^r_A x'$ and $x' \xrightarrow{\Diamond r^* t}_A y$.

The Restricted System $\mathcal{T}_{pat}$

$$\mathcal{R}_{\Diamond} \quad \frac{\Diamond rux}{rxy\,,\;uy} \quad y \notin \mathcal{N}_\iota A$$

- $\mathcal{P}^r_A(\Diamond rux)$ not realized in A
- x is propagated

Request Paths

Definition ($\Diamond^*$-Request Relation)

Let A be a branch and $x, x' \in \mathcal{N}_\iota A$.

$$\begin{aligned} x \xrightarrow{\trianglerighteq, \Diamond r^* t}_A y \iff & \exists \alpha \in \mathcal{V}A\colon \alpha = \Diamond r^* t, \Diamond r \alpha x \in A \\ & \wedge \exists x' \in \mathcal{N}_\iota A\colon \mathcal{R}^r_A x = \mathcal{R}^r_A x' \\ & \wedge x' \xrightarrow{\Diamond r^* t}_A y \end{aligned}$$

Completeness

Supported by two Lemmas. Let A be a maximal branch.

- If a formula $\Diamond r^* tx \in A$ is not evident, then there is a cycle in $\xrightarrow{\trianglerighteq, \Diamond r^* t}_{A}$.
- If A is straight, then no $\Diamond^*$-request relation in A is cyclic.

Explicit Request Relations

$$\mathcal{T}_{\Diamond^*} \; \frac{\alpha = \Diamond r^* t, \Diamond r \alpha x, \beta = \Diamond r^* t, \beta x', r x' y, \beta y}{t x \mid \Diamond r \alpha x} \; \mathcal{R}^r_A x = \mathcal{R}^r_A x'$$