

Formalising the Undecidability of Higher-Order Unification

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Motivation — $\forall n. n + 0 = n$

The screenshot shows a terminal window titled "natind.v" containing a Coq proof script. The script defines a lemma `plus_zero` and provides a proof using `N_ind`. The proof steps are:

- 1 Lemma plus_zero: $\forall n, n + 0 = n$.
- 2 Proof.
- 3 Check `N_ind`.
- 4 apply `N_ind` with ($P := \lambda n \Rightarrow n + 0 = n$).
- 5 reflexivity.
- 6 cbn; now intros ? \rightarrow .
- 7 Qed.

The status bar at the bottom indicates the file is `natind.v` in the `Coq` environment, running on `unix`, with 2:6 All subgoals. The current goal is $\forall n : \mathbb{N}, n + 0 = n$. The bottom tab bar shows the current tab is `*goals*`.

Motivation — $\forall n. n + 0 = n$

The screenshot shows a terminal window titled "natind.v" containing a Coq proof script. The script defines a lemma `plus_zero` and proves it using `N_ind`. The proof steps are:

- 1 Lemma `plus_zero: ∀ n, n + 0 = n.`
- 2 Proof.
- 3 Check `N_ind.`
- 4 apply `N_ind with (P := λ n ⇒ n + 0 = n).`
- 5 reflexivity.
- 6 cbn; now intros ? →.
- 7 Qed.

The command `N_ind` is expanded to its type:

```
: ∀ P : ℙ → ℙ,  
P 0 → (∀ n : ℙ, P n → P (S n)) → ∀ n : ℙ, P n
```

The terminal status bar at the bottom indicates:

- ② * 109 *response* / Coq Response Utoks@e@el / utf-8 | 3:52 Al

Motivation — $\forall n. n + 0 = n$

The screenshot shows a Coq proof assistant interface with the following details:

- File:** natind.v
- Code:**

```
1 Lemma plus_zero: ∀ n, n + 0 = n.
2 Proof.
3   Check N_ind.
4   apply N_ind with (P := λ n ⇒ n + 0 = n).
5   reflexivity.
6   cbn; now intros ? →.
7 Qed.
```
- Status Bar:**
 - ① * 153 natind.v Coq u unix | 5: 0 All
 - 2 subgoals (ID 8)
- Goal:**

```
0 + 0 = 0
```
- Subgoal:**

```
subgoal 2 (ID 9) is:
  ∀ n : N, n + 0 = n → S n + 0 = S n
```
- Bottom Bar:**
 - ~
 - ② % 130 *goals* Coq Goals Utoks@e® utf-8 | 4: 0 All

Motivation — $\forall n. n + 0 = n$

The screenshot shows a terminal window titled "natind.v" containing a Coq proof script. The script defines a lemma `plus_zero` and proves it using induction on `N`. The proof steps are:

- 1 Lemma `plus_zero: ∀ n, n + 0 = n.`
- 2 Proof.
- 3 Check `N_ind.`
- 4 apply `N_ind.`
- 5 reflexivity.
- 6 cbn; now intros ? →.
- 7 Qed.

Below the script, the terminal shows the state of the proof:

- ① * 122 natind.v Coq u unix | 5: 0 All
- 2 subgoals (ID 5)

The first subgoal is displayed as:

$$0 + 0 = 0$$

subgoal 2 (ID 6) is:

$$\forall n : \mathbb{N}, n + 0 = n \rightarrow S n + 0 = S n$$

~

② % 130 *goals* Coq Goals Utoks@e® utf-8 | 4: 0 All

Higher-Order
oooooo

Nth-Order
ooooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Overview



Higher-Order
oooooo

Nth-Order
ooooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Overview

Higher-Order — U

following Dowek (2001)



Overview

Higher-Order — \mathbf{U}

following Dowek (2001)

Nth-Order — \mathbf{U}_n

$$\mathbf{U}_n \subseteq \mathbf{U}$$



Overview

Higher-Order — \mathbf{U}

following Dowek (2001)

Nth-Order — \mathbf{U}_n

$$\mathbf{U}_n \subseteq \mathbf{U}$$

Third-Order — \mathbf{U}_3

following Huet (1973)



Overview

Higher-Order — \mathbf{U}

following Dowek (2001)

Nth-Order — \mathbf{U}_n

$$\mathbf{U}_n \subseteq \mathbf{U}$$

Third-Order — \mathbf{U}_3

following Huet (1973)

Second-Order — \mathbf{U}_2

following Goldfarb (1981)



Higher-Order
●ooooo

Nth-Order
ooooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Example

$$\lambda xy. fx \stackrel{?}{=} \lambda xy. fy$$

Higher-Order
●ooooo

Nth-Order
ooooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Example

$$\lambda xy.\textcolor{red}{fx} \stackrel{?}{=} \lambda xy.\textcolor{red}{fy}$$

Example

$$\Gamma \vdash \lambda xy. \textcolor{red}{f}x \stackrel{?}{=} \lambda xy. \textcolor{red}{f}y : A$$

where $\Gamma = (\textcolor{red}{f} : \alpha \rightarrow \alpha)$ and $A = \alpha \rightarrow \alpha \rightarrow \alpha$.

Example

$$\Gamma \vdash \lambda xy. \textcolor{red}{f}x \stackrel{?}{=} \lambda xy. \textcolor{red}{f}y : A$$

where $\Gamma = (\textcolor{red}{f} : \alpha \rightarrow \alpha)$ and $A = \alpha \rightarrow \alpha \rightarrow \alpha$.

Solution

$$\sigma \textcolor{red}{f} = \lambda _. z$$

$$\sigma x = x \qquad \qquad \text{othw.}$$

Example

$$\Gamma \vdash \lambda xy. \textcolor{red}{f}x \stackrel{?}{=} \lambda xy. \textcolor{red}{f}y : A$$

where $\Gamma = (\textcolor{red}{f} : \alpha \rightarrow \alpha)$ and $A = \alpha \rightarrow \alpha \rightarrow \alpha$.

Solution

$$\sigma \textcolor{red}{f} = \lambda _. z$$

$$\sigma x = x \qquad \text{othw.}$$

in $\Delta = (z : \alpha)$



Example

$$\Gamma \vdash \lambda xy. \textcolor{red}{f}x \stackrel{?}{=} \lambda xy. \textcolor{red}{f}y : A$$

where $\Gamma = (\textcolor{red}{f} : \alpha \rightarrow \alpha)$ and $A = \alpha \rightarrow \alpha \rightarrow \alpha$.

Solution

$$\sigma \textcolor{red}{f} = \lambda _. z$$

$$\sigma x = x$$

in $\Delta = (z : \alpha)$

Proof

$$(\lambda xy. \textcolor{red}{f}x)[\sigma]$$

othw.



Example

$$\Gamma \vdash \lambda xy.\textcolor{red}{f}x \stackrel{?}{=} \lambda xy.\textcolor{red}{f}y : A$$

where $\Gamma = (\textcolor{red}{f} : \alpha \rightarrow \alpha)$ and $A = \alpha \rightarrow \alpha \rightarrow \alpha$.

Solution

$$\sigma \textcolor{red}{f} = \lambda _.z$$

$$\sigma x = x$$

in $\Delta = (z : \alpha)$

Proof

$$(\lambda xy.\textcolor{red}{f}x)[\sigma] \equiv \lambda xy.z$$

othw.



Example

$$\Gamma \vdash \lambda xy.\textcolor{red}{f}x \stackrel{?}{=} \lambda xy.\textcolor{red}{f}y : A$$

where $\Gamma = (\textcolor{red}{f} : \alpha \rightarrow \alpha)$ and $A = \alpha \rightarrow \alpha \rightarrow \alpha$.

Solution

$$\sigma \textcolor{red}{f} = \lambda _.z$$

$$\sigma x = x$$

in $\Delta = (z : \alpha)$

Proof

$$(\lambda xy.\textcolor{red}{f}x)[\sigma] \equiv \lambda xy.z$$

othw.

$$\equiv (\lambda xy.\textcolor{red}{f}y)[\sigma]$$



Higher-Order
○●○○○

Nth-Order
○○○○○

Third-Order
○○○○

Second-Order
○○○○○○

Conclusion
○○○

Higher-Order Unification — U

$$\mathbf{U} \ (\Gamma \vdash s \stackrel{?}{=} t : A)$$



Higher-Order Unification — U

$$\mathbf{U} (\Gamma \vdash s \stackrel{?}{=} t : A) := \exists \sigma \quad s[\sigma] \equiv t[\sigma]$$

$$s \equiv t$$

$$\frac{s \succ^* v \quad t \succ^* v}{s \equiv t}$$

Higher-Order Unification — U

$$\mathbf{U} (\Gamma \vdash s \stackrel{?}{=} t : A) := \\ \exists \sigma \Delta. \Delta \vdash \sigma : \Gamma \quad \text{and} \quad s[\sigma] \equiv t[\sigma]$$

$$\boxed{\Delta \vdash \sigma : \Gamma}$$

$$\boxed{s \equiv t}$$

$$\frac{\forall(x:A) \in \Gamma. \Delta \vdash \sigma x : A}{\Delta \vdash \sigma : \Gamma}$$

$$\frac{s \succ^* v \quad t \succ^* v}{s \equiv t}$$



Higher-Order
○○●○○

Nth-Order
○○○○○

Third-Order
○○○○

Second-Order
○○○○○○

Conclusion
○○○

Undecidability

H10 ↳ U

Higher-Order
○○●○○○

Nth-Order
○○○○○

Third-Order
○○○○

Second-Order
○○○○○○

Conclusion
○○○

Undecidability

H10 ↴ **SU** ↴ **U**



Undecidability

H10 \preceq **SU** \preceq **U**

SU ($\{\Gamma \vdash s_i \stackrel{?}{=} t_i : A_i \mid i = 1, \dots, n\}$) :=
 $\exists \sigma \Delta. \Delta \vdash \sigma : \Gamma \quad \text{and} \quad \forall i. s_i[\sigma] \equiv t_i[\sigma]$

Hilbert's tenth problem — H10

Example

$$x \doteq 42$$

$$y \doteq x \cdot y$$

$$z \doteq z + z$$



Hilbert's tenth problem — H10

Example

$$x \doteq 42 \quad y \doteq x \cdot y \quad z \doteq z + z$$

Solution

$$\theta x = 42 \quad \theta y = \theta z = 0$$



Hilbert's tenth problem — H10

Example

$$x \doteq 42$$

$$y \doteq x \cdot y$$

$$z \doteq z + z$$

Solution

$$\theta x = 42$$

$$\theta y = \theta z = 0$$

$$\boxed{\theta \vDash d}$$

$$\theta \vDash x \doteq c \quad \text{iff} \quad \theta x = c$$

$$\theta \vDash x + y \doteq z \quad \text{iff} \quad \theta y + \theta y = \theta z$$

$$\theta \vDash x \cdot y \doteq z \quad \text{iff} \quad \theta y \cdot \theta y = \theta z$$



Hilbert's tenth problem — H10

Example

$$x \doteq 42 \quad y \doteq x \cdot y \quad z \doteq z + z$$

Solution

$$\theta x = 42 \quad \theta y = \theta z = 0$$

$$\boxed{\theta \vDash d}$$

$$\theta \vDash x \doteq c \quad \text{iff} \quad \theta x = c$$

$$\theta \vDash x + y \doteq z \quad \text{iff} \quad \theta y + \theta y = \theta z$$

$$\theta \vDash x \cdot y \doteq z \quad \text{iff} \quad \theta y \cdot \theta y = \theta z$$

$$\mathbf{H10}(D) := \exists \theta. \forall d \in D. \theta \vDash d$$



Higher-Order
oooo●o

Nth-Order
ooooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Church Numerals

$$[\![n]\!]$$
$$[\![n]\!] := \lambda af.f^n\ a$$


Higher-Order
oooo●○

Nth-Order
oooo○

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Church Numerals

$\llbracket n \rrbracket$

$\llbracket n \rrbracket := \lambda a f. f^n a$

Operations

$\text{add } s t := \lambda a f. s (t a f) f$ $\text{mul } s t := \lambda a f. s a (\lambda b. t b f)$



Church Numerals

 $\llbracket n \rrbracket$

$$\llbracket n \rrbracket := \lambda a f. f^n a$$

Operations

$$\text{add } s t := \lambda a f. s (t a f) f \quad \text{mul } s t := \lambda a f. s a (\lambda b. t b f)$$

Characteristic Equation $f^n(fa) = f(f^n a)$



Church Numerals

 $\llbracket n \rrbracket$

$$\llbracket n \rrbracket := \lambda a f. f^n a$$

Operations

$$\text{add } s t := \lambda a f. s (t a f) f \quad \text{mul } s t := \lambda a f. s a (\lambda b. t b f)$$

Characteristic Equation Let s be a normal.

$$\lambda a f. s (f a) f \equiv \lambda a f. f (s a f) \quad \text{iff} \quad s = \llbracket n \rrbracket \quad \text{for some } n : \mathbb{N}$$



Higher-Order
oooooo

Nth-Order
ooooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

H10 \preceq **SU**

H10 \preceq **SU**



Higher-Order
oooooo

Nth-Order
ooooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

H10 \preceq **SU**

H10(D) iff **SU**(\overline{D})



H10 \preceq **SU**

$$\mathbf{H10}(D) \quad \text{iff} \quad \mathbf{SU}(\overline{D})$$

Proof.

Pick \overline{D} :

$$\overline{x \doteq c} := x \stackrel{?}{=} \llbracket c \rrbracket \qquad \overline{x + y \doteq z} := \text{add } x \; y \stackrel{?}{=} z$$

$$\overline{x \cdot y \doteq z} := \text{mul } x \; y \stackrel{?}{=} z$$



H10 \preceq SU

$$\mathbf{H10}(D) \quad \text{iff} \quad \mathbf{SU}(\overline{D})$$

Proof.

Pick \overline{D} :

$$\overline{x \doteq c} := x \stackrel{?}{=} \llbracket c \rrbracket \quad \overline{x + y \doteq z} := \mathbf{add}\ x\ y \stackrel{?}{=} z$$

$$\overline{x \cdot y \doteq z} := \mathbf{mul}\ x\ y \stackrel{?}{=} z$$

$$\overline{x} := \lambda a f. x\ (f\ a) \ f \stackrel{?}{=} \lambda a f. f\ (x\ a\ f)$$



Overview

Higher-Order — \mathbf{U}

following Dowek (2001)

Nth-Order — \mathbf{U}_n

$$\mathbf{U}_n \subseteq \mathbf{U}$$

Third-Order — \mathbf{U}_3

following Huet (1973)

Second-Order — \mathbf{U}_2

following Goldfarb (1981)

Higher-Order
oooooo

Nth-Order
o•ooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Order

First-Order

α

β

γ

δ

...



Higher-Order
oooooo

Nth-Order
o•ooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Order

First-Order

α

β

γ

δ

...

Second-Order

$\alpha \rightarrow \alpha$

$\beta \rightarrow \alpha$

$\alpha \rightarrow \beta \rightarrow \gamma$



Higher-Order
oooooo

Nth-Order
○●ooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Order

First-Order

α

β

γ

δ

...

Second-Order

$\alpha \rightarrow \alpha$

$\beta \rightarrow \alpha$

$\alpha \rightarrow \beta \rightarrow \gamma$

Third-Order

$(\alpha \rightarrow \alpha) \rightarrow \alpha$

$(\alpha \rightarrow \beta \rightarrow \alpha) \rightarrow \beta$

$(\beta \rightarrow \alpha) \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma$



Higher-Order
oooooo

Nth-Order
o•ooo

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Order

ord A

$$\text{ord } \alpha = 1 \quad \text{ord } (A \rightarrow B) = \max\{\text{ord } A + 1, \text{ord } B\}$$

First-Order

α

β

γ

δ

...

Second-Order

$\alpha \rightarrow \alpha$

$\beta \rightarrow \alpha$

$\alpha \rightarrow \beta \rightarrow \gamma$

Third-Order

$(\alpha \rightarrow \alpha) \rightarrow \alpha$

$(\alpha \rightarrow \beta \rightarrow \alpha) \rightarrow \beta$

$(\beta \rightarrow \alpha) \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma$



Nth-Order Fragment

$$\boxed{\Gamma \vdash_n s : A}$$

Let Ω be a signature.

$$\frac{(x : A) \in \Gamma \quad \text{ord } A \leq n}{\Gamma \vdash_n x : A}$$

$$\frac{\text{ord } (\Omega c) \leq n + 1}{\Gamma \vdash_n c : \Omega c}$$

$$\frac{\Gamma \vdash_n s : A \rightarrow B \quad \Gamma \vdash_n t : A}{\Gamma \vdash_n s \ t : B}$$

$$\frac{\Gamma, x : A \vdash_n s : B}{\Gamma \vdash_n \lambda x. s : A \rightarrow B}$$

Nth-Order Fragment

$$\boxed{\Gamma \vdash_n s : A}$$

Let Ω be a signature.

$$\frac{(x : A) \in \Gamma \quad \text{ord } A \leq n}{\Gamma \vdash_n x : A}$$

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$$\frac{\Gamma \vdash_n s : A \rightarrow B \quad \Gamma \vdash_n t : A}{\Gamma \vdash_n s \ t : B}$$

$$\frac{\Gamma, x : A \vdash_n s : B}{\Gamma \vdash_n \lambda x. s : A \rightarrow B}$$

Examples

$$\Gamma \vdash_1 \lambda x. x : \alpha \rightarrow \alpha$$

Nth-Order Fragment

$$\boxed{\Gamma \vdash_n s : A}$$

Let Ω be a signature.

$$\frac{(x : A) \in \Gamma \quad \text{ord } A \leq n}{\Gamma \vdash_n x : A}$$

$$\frac{\text{ord } (\Omega c) \leq n + 1}{\Gamma \vdash_n c : \Omega c}$$

$$\frac{\Gamma \vdash_n s : A \rightarrow B \quad \Gamma \vdash_n t : A}{\Gamma \vdash_n s \ t : B}$$

$$\frac{\Gamma, x : A \vdash_n s : B}{\Gamma \vdash_n \lambda x. s : A \rightarrow B}$$

Examples

$$\Gamma \vdash_1 \lambda x. x : \alpha \rightarrow \alpha$$

$$\Gamma \vdash_2 \lambda x. x : \alpha \rightarrow \alpha$$

Nth-Order Fragment

$$\boxed{\Gamma \vdash_n s : A}$$

Let Ω be a signature.

$$\frac{(x : A) \in \Gamma \quad \text{ord } A \leq n}{\Gamma \vdash_n x : A}$$

$$\frac{\text{ord } (\Omega c) \leq n + 1}{\Gamma \vdash_n c : \Omega c}$$

$$\frac{\Gamma \vdash_n s : A \rightarrow B \quad \Gamma \vdash_n t : A}{\Gamma \vdash_n s \ t : B}$$

$$\frac{\Gamma, x : A \vdash_n s : B}{\Gamma \vdash_n \lambda x. s : A \rightarrow B}$$

Examples

$$\Gamma \vdash_1 \lambda x. x : \alpha \rightarrow \alpha$$

$$\Gamma \vdash_2 \lambda x. x : \alpha \rightarrow \alpha$$

$$\Gamma \vdash_3 \lambda x. x : ((\alpha \rightarrow \alpha) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha) \rightarrow \alpha$$

Nth-Order Unification — \mathbf{U}_n

$\mathbf{U}_n \ (\Gamma \vdash_n s \stackrel{?}{=} t : A) :=$
 $\exists \sigma \Delta. \Delta \vdash_n \sigma : \Gamma \quad \text{and} \quad s[\sigma] \equiv t[\sigma]$

$$\boxed{\Delta \vdash_n \sigma : \Gamma}$$

$$\frac{\forall(x : A) \in \Gamma. \Delta \vdash_n \sigma x : A}{\Delta \vdash_n \sigma : \Gamma}$$

Higher-Order
oooooo

Nth-Order
oooo●

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Conservativity

$$\mathbf{PCP} \preceq \mathbf{U}$$

Higher-Order
oooooo

Nth-Order
oooo●

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Conservativity

PCP \preceq **U**₃ and **PCP** \preceq **U**

Higher-Order
oooooo

Nth-Order
oooo●

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Conservativity

$$\mathbf{PCP} \preceq \mathbf{U}_3 \quad \text{and} \quad \mathbf{PCP} \preceq \mathbf{U} \quad \leadsto \quad \mathbf{PCP} \preceq \mathbf{U}_3 \preceq \mathbf{U}$$

Conservativity

$$\mathbf{PCP} \preceq \mathbf{U}_3 \quad \text{and} \quad \mathbf{PCP} \preceq \mathbf{U} \quad \leadsto \quad \mathbf{PCP} \preceq \mathbf{U}_3 \preceq \mathbf{U}$$

Conservativity Let $n \leq m$.

$$\mathbf{U}_n \quad \preceq \quad \mathbf{U}_m \quad \preceq \quad \mathbf{U}$$

Higher-Order
oooooo

Nth-Order
oooo●

Third-Order
oooo

Second-Order
oooooo

Conclusion
ooo

Conservativity

$$\mathbf{PCP} \preceq \mathbf{U}_3 \quad \text{and} \quad \mathbf{PCP} \preceq \mathbf{U} \quad \leadsto \quad \mathbf{PCP} \preceq \mathbf{U}_3 \preceq \mathbf{U}$$

Conservativity Let $n \leq m$.

$$\mathbf{U}_n \quad \subseteq \quad \mathbf{U}_m \quad \subseteq \quad \mathbf{U}$$

Overview

Higher-Order — \mathbf{U}

following Dowek (2001)

Nth-Order — \mathbf{U}_n

$$\mathbf{U}_n \subseteq \mathbf{U}$$

Third-Order — \mathbf{U}_3

following Huet (1973)

Second-Order — \mathbf{U}_2

following Goldfarb (1981)

Third-Order Unification — \mathbf{U}_3

Huet (1973)

$\mathbf{PCP} \preceq \mathbf{U}_3$

Third-Order Unification — \mathbf{U}_3

Huet (1973)

$$\mathbf{PCP} \preceq \mathbf{U}_3$$



This Work

$$\mathbf{MPCP} \preceq \mathbf{U}_3$$

Modified Post Correspondence Problem — MPCP

Given

$$\begin{array}{r} 11 \\ \hline 1 \end{array}$$

and

$$\begin{array}{r} 0 \\ \hline 110 \end{array}$$

$$\begin{array}{r} 101 \\ \hline 000 \end{array}$$

①

②

③

Modified Post Correspondence Problem — MPCP

Given

$$\begin{array}{c} 11 \\ \hline 1 \end{array}$$

and

$$\begin{array}{c} 0 \\ \hline 110 \end{array}$$

$$\begin{array}{c} 101 \\ \hline 000 \end{array}$$

①

②

③

Find Ordering

① ②

Such that

$$\begin{array}{|c|c|c|} \hline & \begin{array}{c} 11 \\ \hline 1 \end{array} & \begin{array}{c} 11 \\ \hline 1 \end{array} & \begin{array}{c} 0 \\ \hline 110 \end{array} \\ \hline \end{array}$$

①

②

Modified Post Correspondence Problem — MPCP

Given

$$\begin{array}{|c|}\hline 11 \\ \hline 1 \\ \hline \end{array}$$

and

$$\begin{array}{|c|}\hline 0 \\ \hline 110 \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline 101 \\ \hline 000 \\ \hline \end{array}$$

(0)

(1)

(2)

Find Ordering

(0) (1)

Such that

$$\begin{array}{|c|c|c|}\hline 11 & 11 & 0 \\ \hline 1 & 1 & 110 \\ \hline \end{array} = \begin{array}{|c|}\hline 11110 \\ \hline 11110 \\ \hline \end{array}$$

(0) (1)

Modified Post Correspondence Problem — MPCP

Given

$$\boxed{\frac{l_0}{r_0}}$$

and

$$\boxed{\frac{l_1}{r_1}}$$

...

$$\boxed{\frac{l_n}{r_n}}$$

①

②

③

Find Ordering

$$i_1, \dots, i_k$$

Such that

$$l_0 l_{i_1} \cdots l_{i_k} = r_0 r_{i_1} \cdots r_{i_k}$$

Higher-Order
oooooo

Nth-Order
ooooo

Third-Order
ooo●

Second-Order
oooooo

Conclusion
ooo

Reduction

$$(\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$$

Reduction

$$f \ g_0 \ \cdots \ g_n \ s \ \equiv \ g_{i_1} (\cdots (g_{i_k} s))$$

where $f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

Reduction

$$f \ g_0 \ \cdots \ g_n \ s \ \equiv \ g_{i_1} (\cdots (g_{i_k} s))$$

where $f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

Reduction

$$f \ g_0 \ \cdots \ g_n \ s \ \equiv \ g_{i_1} (\cdots (g_{i_k} s))$$

where $f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

Encoding Fix $u_1, u_0 : \alpha \rightarrow \alpha$.

Reduction

$$f \ g_0 \ \cdots \ g_n \ s \ \equiv \ g_{i_1} (\cdots (g_{i_k} s))$$

where $f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

Encoding Fix $u_1, u_0 : \alpha \rightarrow \alpha$.

$$\overline{110} := \lambda x. u_1 (u_1 (u_0 x))$$

Reduction

$$f \ \overline{l_0} \ \cdots \ \overline{l_n} \ \epsilon \ \equiv \ \overline{l_{i_1}} \ (\cdots (\overline{l_{i_k}} \ \epsilon))$$

where $f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

Encoding Fix $u_1, u_0 : \alpha \rightarrow \alpha$.

$$\overline{110} := \lambda x. u_1 \ (u_1 \ (u_0 \ x))$$

Reduction

$$f \ \overline{l_0} \ \cdots \ \overline{l_n} \ \epsilon \ \equiv \ \overline{l_{i_1}} \ (\cdots (\overline{l_{i_k}} \ \epsilon))$$

where $f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

Encoding Fix $u_1, u_0 : \alpha \rightarrow \alpha$.

$$\overline{110} := \lambda x. u_1 \ (u_1 \ (u_0 \ x))$$

$$\bar{l} \ (\overline{l'} \ s) \equiv \overline{ll'} \ s \quad \text{and} \quad \overline{l_{i_1}} \ (\cdots (\overline{l_{i_k}} \ s)) \equiv \overline{l_{i_1} \cdots l_{i_k}} \ s$$

Reduction

$$f \ \overline{l_0} \ \cdots \ \overline{l_n} \ \epsilon \ \equiv \ \overline{l_{i_1} \cdots l_{i_k}} \ \epsilon$$

where $f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

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Reduction

$$f \ \overline{r_0} \ \cdots \ \overline{r_n} \ \epsilon \ \equiv \ \overline{r_{i_1} \cdots r_{i_k}} \ \epsilon$$

where $f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

Encoding Fix $u_1, u_0 : \alpha \rightarrow \alpha$.

$$\overline{110} := \lambda x. u_1 \ (u_1 \ (u_0 \ x))$$

$$\bar{l} \ (\bar{l'} \ s) \equiv \bar{ll'} \ s \quad \text{and} \quad \overline{l_{i_1}} \ (\cdots (\overline{l_{i_k}} \ s)) \equiv \overline{l_{i_1} \cdots l_{i_k}} \ s$$

Reduction

$$x_f \ \overline{l_0} \ \cdots \ \overline{l_n} \ \epsilon \stackrel{?}{=} x_f \ \overline{r_0} \ \cdots \ \overline{r_n} \ \epsilon$$

where $x_f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

Encoding Fix $u_1, u_0 : \alpha \rightarrow \alpha$.

$$\overline{110} := \lambda x. u_1 \ (u_1 \ (u_0 \ x))$$

$$\bar{l} \ (\bar{l}' \ s) \equiv \bar{l} \bar{l}' \ s \quad \text{and} \quad \overline{l_{i_1}} \ (\cdots (\overline{l_{i_k}} \ s)) \equiv \overline{l_{i_1} \cdots l_{i_k}} \ s$$

Reduction

$$\lambda u_1 u_0. x_f \overline{l_0} \cdots \overline{l_n} \epsilon \stackrel{?}{=} \lambda u_1 u_0. x_f \overline{r_0} \cdots \overline{r_n} \epsilon$$

where $x_f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

Encoding Fix $u_1, u_0 : \alpha \rightarrow \alpha$.

$$\overline{110} := \lambda x. u_1 (u_1 (u_0 x))$$

$$\bar{l} (\bar{l'} s) \equiv \overline{ll'} s \quad \text{and} \quad \overline{l_{i_1}} (\cdots (\overline{l_{i_k}} s)) \equiv \overline{l_{i_1} \cdots l_{i_k}} s$$

Reduction

$$\lambda u_1 u_0. \overline{\textcolor{teal}{l}_0} (x_f \ \overline{l_0} \ \cdots \ \overline{l_n} \ \epsilon) \stackrel{?}{=} \lambda u_1 u_0. \overline{\textcolor{teal}{r}_0} (x_f \ \overline{r_0} \ \cdots \ \overline{r_n} \ \epsilon)$$

where $x_f : (\alpha \rightarrow \alpha)^{n+1} \rightarrow \alpha \rightarrow \alpha$

Encoding Fix $u_1, u_0 : \alpha \rightarrow \alpha$.

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Reduction

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Encoding Fix $u_1, u_0 : \alpha \rightarrow \alpha$.

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$$\bar{l} (\bar{l'} s) \equiv \bar{l l'} s \quad \text{and} \quad \overline{l_{i_1}} (\cdots (\overline{l_{i_k}} s)) \equiv \overline{l_{i_1} \cdots l_{i_k}} s$$

Reduction

$$\lambda u_1 u_0. \overline{l_0} (x_f \ \overline{l_0} \ \cdots \ \overline{l_n}) \stackrel{?}{=} \lambda u_1 u_0. \overline{r_0} (x_f \ \overline{r_0} \ \cdots \ \overline{r_n})$$

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Overview

Higher-Order — \mathbf{U}

following Dowek (2001)

Nth-Order — \mathbf{U}_n

$$\mathbf{U}_n \subseteq \mathbf{U}$$

Third-Order — \mathbf{U}_3

following Huet (1973)

Second-Order — \mathbf{U}_2

following Goldfarb (1981)

Undecidability Second-Order

Goldfarb's Result

$$\mathsf{H10} \preceq \mathbf{U}_2^{\{g,a,b\}}$$

where $\mathbf{U}_2^{\{g,a,b\}}$ is second-order unification with constants
 $g : \alpha \rightarrow \alpha \rightarrow \alpha$ and $a, b : \alpha$.

Goldfarb Numerals

 $\llbracket n \rrbracket$

$$\llbracket n \rrbracket := \lambda a f. f^n a$$

Operations

$$\text{add } s t := \lambda a f. s (t a f) f \quad \text{mul } s t := \lambda a f. s a (\lambda b. t b f)$$

Characteristic Equation Let s be a normal.

$$\lambda a f. s (f a) f \equiv \lambda a f. f (s a f) \quad \text{iff} \quad s = \llbracket n \rrbracket \quad \text{for some } n : \mathbb{N}$$

Goldfarb Numerals

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Goldfarb Numerals

 $\llbracket n \rrbracket$

$$\llbracket n \rrbracket := \lambda a. (\text{g } a)^n \ a$$

Operations

$$\text{add } s \ t := \lambda a f. s \ (t \ a \ f) \ f \quad \text{mul } s \ t := \lambda a f. s \ a \ (\lambda b. t \ b \ f)$$

Characteristic Equation Let s be a normal.

$$\lambda a f. s \ (f \ a) \ f \equiv \lambda a f. f \ (s \ a \ f) \quad \text{iff} \quad s = \llbracket n \rrbracket \quad \text{for some } n : \mathbb{N}$$

Goldfarb Numerals

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Goldfarb Numerals

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Goldfarb Numerals

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Characteristic Equation Let s be a normal.

$$\lambda a f. s\ (f\ a)\ f \equiv \lambda a f. f\ (s\ a\ f) \quad \text{iff} \quad s = \llbracket n \rrbracket \quad \text{for some } n : \mathbb{N}$$

Goldfarb Numerals

 $\llbracket n \rrbracket$

$$\llbracket n \rrbracket := \lambda a. (g\ a)^n\ a$$

Operations

$$\text{add } s\ t := \lambda a.s\ (t\ a) \qquad \text{mul } s\ t := \lambda a f. s\ a\ (\lambda b. t\ b\ f)$$

Characteristic Equation Let s be a normal.

$$\lambda a. s\ (\textcolor{teal}{f}\ a) \equiv \lambda a. \textcolor{teal}{f}\ (s\ a) \quad \text{iff} \quad s = \llbracket n \rrbracket \quad \text{for some } n : \mathbb{N}$$

Goldfarb Numerals

 $\llbracket n \rrbracket$

$$\llbracket n \rrbracket := \lambda a. (g\ a)^n\ a$$

Operations

$$\text{add } s\ t := \lambda a. s\ (t\ a) \qquad \text{mul } s\ t := \lambda a f. s\ a\ (\lambda b. t\ b\ f)$$

Characteristic Equation Let s be a normal.

$$\lambda a. s\ ((g\ a)\ a) \equiv \lambda a. (g\ a)\ (s\ a) \quad \text{iff} \quad s = \llbracket n \rrbracket \quad \text{for some } n : \mathbb{N}$$

Goldfarb Numerals

 $\llbracket n \rrbracket$

$$\llbracket n \rrbracket := \lambda a. (g\ a)^n\ a$$

Operations

$$\text{add } s\ t := \lambda a. s\ (t\ a) \quad \text{mul } s\ t := \lambda a f. s\ a\ (\lambda b. t\ b\ f)$$

Characteristic Equation Let s be a normal.

$$\lambda a. s\ ((g\ a)\ a) \equiv \lambda a. (g\ a)\ (s\ a) \quad \text{iff} \quad \forall t. s\ t \equiv \llbracket n \rrbracket t \text{ for some } n : \mathbb{N}$$



Goldfarb Numerals

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Goldfarb Numerals

 $\llbracket n \rrbracket$

$$\llbracket n \rrbracket := \lambda a. (g\ a)^n\ a$$

Operations

$$\text{add } s\ t := \lambda a. s\ (t\ a)$$

$$\text{mul } s\ t := ???$$

Characteristic Equation Let s be a normal.

$$\lambda a. s\ ((g\ a)\ a) \equiv \lambda a. (g\ a)\ (s\ a) \quad \text{iff} \quad \forall t. s\ t \equiv \llbracket n \rrbracket\ t \text{ for some } n : \mathbb{N}$$



Multiplication — $m \cdot n = p$

$\text{mult}(0, 0)$ where

$$\text{mult}(a, i) = a$$

if $i = m$

$$\text{mult}(a, i) = \text{mult}(a + n, i + 1)$$

if $i \neq m$



Higher-Order
oooooo

Nth-Order
ooooo

Third-Order
oooo

Second-Order
ooo•ooo

Conclusion
ooo

Multiplication — $m \cdot n = p$

$\text{mult}(0, 0)$ where

$\text{mult}(a, i) = a$ if $i = m$

$\text{mult}(a, i) = \text{mult}(a + n, i + 1)$ if $i \neq m$

Multiplication Sequence

$$(0, 0); (n, 1); (2n, 2); \dots; (m \cdot n, m)$$



Multiplication — $m \cdot n = p$

`mult(0, 0)` where

`mult(a, i) = a` if $i = m$

`mult(a, i) = mult(a + n, i + 1)` if $i \neq m$

Multiplication Sequence

$$(0, 0); (n, 1); (2n, 2); \dots; (\textcolor{teal}{m} \cdot \textcolor{teal}{n}, m)$$

Higher-Order
oooooo

Nth-Order
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Third-Order
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Second-Order
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Conclusion
ooo

Multiplication — $m \cdot n = p$

$\text{mult}(0, 0)$ where

$\text{mult}(a, i) = a$ if $i = m$

$\text{mult}(a, i) = \text{mult}(a + n, i + 1)$ if $i \neq m$

Multiplication Sequence

$$(0, 0); (n, 1); (2n, 2); \dots; (\textcolor{teal}{p}, m)$$



Multiplication — $m \cdot n = p$

$$m \cdot n = p \quad \text{iff} \quad \exists X. (0, 0); \text{succ } X = X; (p, m)$$

where $\text{succ}(a, i) := (a + n, i + 1)$
 $\text{succ } X := \text{map succ } X$

Multiplication — $m \cdot n = p$

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(0, 0)	$\text{succ } X$	
X		(p, m)

Multiplication — $m \cdot n = p$

$$m \cdot n = p \quad \text{iff} \quad \exists X. (0, 0); \text{succ } X = X; (p, m)$$

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$\text{succ } X := \text{map succ } X$

t_0	$\text{succ } X$	
	X	(p, m)

where $t_i := (i \cdot n, i)$ and $\text{succ}(t_i) = t_{i+1}$.



Multiplication — $m \cdot n = p$

$$m \cdot n = p \quad \text{iff} \quad \exists X. (0, 0); \text{succ } X = X; (p, m)$$

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$\text{succ } X := \text{map succ } X$

t_0	$\text{succ } t_0$	$\text{succ } X$
t_0	X	(p, m)

where $t_i := (i \cdot n, i)$ and $\text{succ}(t_i) = t_{i+1}$.

Multiplication — $m \cdot n = p$

$$m \cdot n = p \quad \text{iff} \quad \exists X. (0, 0); \text{succ } X = X; (p, m)$$

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$\text{succ } X := \text{map succ } X$

t_0	t_1	$\text{succ } X$
t_0	X	(p, m)

where $t_i := (i \cdot n, i)$ and $\text{succ}(t_i) = t_{i+1}$.

Multiplication — $m \cdot n = p$

$$m \cdot n = p \quad \text{iff} \quad \exists X. (0, 0); \text{succ } X = X; (p, m)$$

where $\text{succ}(a, i) := (a + n, i + 1)$
 $\text{succ } X := \text{map succ } X$

t_0	t_1	t_2	$\text{succ } X$	
t_0	t_1		X	(p, m)

where $t_i := (i \cdot n, i)$ and $\text{succ}(t_i) = t_{i+1}$.

Multiplication — $m \cdot n = p$

$$m \cdot n = p \quad \text{iff} \quad \exists X. (0, 0); \text{succ } X = X; (p, m)$$

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 $\text{succ } X := \text{map succ } X$

t_0	t_1	t_2	\dots	$\text{succ } X$
t_0	t_1	\dots	X	(p, m)

where $t_i := (i \cdot n, i)$ and $\text{succ}(t_i) = t_{i+1}$.

Multiplication — $m \cdot n = p$

$$m \cdot n = p \quad \text{iff} \quad \exists X. (0, 0); \text{succ } X = X; (p, m)$$

where $\text{succ}(a, i) := (a + n, i + 1)$

$\text{succ } X := \text{map succ } X$

t_0	t_1	t_2	\dots	t_m
t_0	t_1	\dots	t_{m-1}	(p, m)

where $t_i := (i \cdot n, i)$ and $\text{succ}(t_i) = t_{i+1}$.

Multiplication — $m \cdot n = p$

$$m \cdot n = p \quad \text{iff} \quad \exists X. (0, 0); \text{succ } X = X; (p, m)$$

where $\text{succ}(a, i) := (a + n, i + 1)$

$\text{succ } X := \text{map succ } X$

t_0	t_1	t_2	\dots	$(m \cdot n, m)$
t_0	t_1	\dots	t_{m-1}	(p, m)

where $t_i := (i \cdot n, i)$ and $\text{succ}(t_i) = t_{i+1}$.



Multiplication Equations

$(0, 0); \text{succ } X = X; (p, m)$

Multiplication Equations

$$(0, 0); \text{succ } X \stackrel{?}{=} X; (z, x)$$

Multiplication Equations

$$(\llbracket 0 \rrbracket, \llbracket 0 \rrbracket) :: \text{succ } (X \ \llbracket \rrbracket) \stackrel{?}{=} X \ [(z, x)]$$

$$(s, t) := g \ s \ t \quad s :: t := g \ s \ t \quad \llbracket \rrbracket := a$$

Multiplication Equations

$$([0], [0]) :: X \ y \ [1] \ [] \stackrel{?}{=} X \ [0] \ [0] \ [(z, x)]$$

$$(s, t) := g \ s \ t \quad s :: t := g \ s \ t \quad [] := a$$

Multiplication Equations

$$(\llbracket 0 \rrbracket a, \llbracket 0 \rrbracket b) :: X (ya) (\llbracket 1 \rrbracket b) [] \stackrel{?}{=} X (\llbracket 0 \rrbracket a) (\llbracket 0 \rrbracket b) [(za, xb)]$$

$$(s, t) := g\ s\ t \qquad s :: t := g\ s\ t \qquad [] := a$$

Multiplication Equations

$$(\llbracket 0 \rrbracket a, \llbracket 0 \rrbracket b) :: X (ya) (\llbracket 1 \rrbracket b) [] \stackrel{?}{=} X (\llbracket 0 \rrbracket a) (\llbracket 0 \rrbracket b) [(za, xb)]$$

$$(\llbracket 0 \rrbracket b, \llbracket 0 \rrbracket a) :: X (yb) (\llbracket 1 \rrbracket a) [] \stackrel{?}{=} X (\llbracket 0 \rrbracket b) (\llbracket 0 \rrbracket a) [(zb, xa)]$$

$$(s, t) := g\ s\ t \qquad s :: t := g\ s\ t \qquad [] := a$$

Higher-Order
oooooo

Nth-Order
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Third-Order
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Second-Order
oooooo

Conclusion
●ooo

Contributions



Higher-Order
oooooo

Nth-Order
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Third-Order
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Second-Order
oooooo

Conclusion
●ooo

Contributions

Higher-Order

$$H10 \preceq SU \preceq U$$



Higher-Order
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Nth-Order
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Third-Order
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Second-Order
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Conclusion
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Contributions

Higher-Order

$$H10 \preceq SU \preceq U$$

Nth-Order

$$U_n \subseteq U$$



Higher-Order
oooooo

Nth-Order
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Third-Order
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Second-Order
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Conclusion
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Contributions

Higher-Order

$$H10 \preceq SU \preceq U$$

Nth-Order

$$U_n \subseteq U$$

Third-Order

$$MPCP \preceq U_3$$



Higher-Order
oooooo

Nth-Order
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Third-Order
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Second-Order
oooooo

Conclusion
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Contributions

Higher-Order

$$H10 \preceq SU \preceq U$$

Third-Order

$$MPCP \preceq U_3$$

Nth-Order

$$U_n \subseteq U$$

Second-Order

$$H10 \preceq U_2^{\{g,a,b\}}$$

Furthermore. . .

- Adding and Removing Constants

$$\mathbf{U}_2^{\{g,a,b\}} \prec \mathbf{U}_2^{\{g\}} \prec \mathbf{U}_3^{\{g\}} \prec \mathbf{U}_3^{\emptyset} \prec \mathbf{U}_3$$

Furthermore. . .

- Adding and Removing Constants

$$\mathbf{U}_2^{\{g,a,b\}} \prec \mathbf{U}_2^{\{g\}} \prec \mathbf{U}_3^{\{g\}} \prec \mathbf{U}_3^{\emptyset} \prec \mathbf{U}_3$$

- First-Order Unification

\mathbf{U}_1 is decidable

Furthermore. . .

- Adding and Removing Constants

$$\mathbf{U}_2^{\{g,a,b\}} \preceq \mathbf{U}_2^{\{g\}} \preceq \mathbf{U}_3^{\{g\}} \preceq \mathbf{U}_3^{\emptyset} \preceq \mathbf{U}_3$$

- First-Order Unification

\mathbf{U}_1 is decidable

- Enumerability

$\mathbf{U}, \mathbf{SU}, \mathbf{U}_n$, and \mathbf{SU}_n are enumerable

Furthermore. . .

- Adding and Removing Constants

$$\mathbf{U}_2^{\{g,a,b\}} \preceq \mathbf{U}_2^{\{g\}} \preceq \mathbf{U}_3^{\{g\}} \preceq \mathbf{U}_3^{\emptyset} \preceq \mathbf{U}_3$$

- First-Order Unification

\mathbf{U}_1 is decidable

- Enumerability

$\mathbf{U}, \mathbf{SU}, \mathbf{U}_n$, and \mathbf{SU}_n are enumerable

Future Work

- Decidability Monadic Second-Order Unification
- Huet's Unification Procedure



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Formalisation

Overview	Spec	Proofs
λ -calculus	790	1120
Unification	350	380
Third-Order	190	400
Second-Order	570	850
First-Order	290	510
Convervativity & Constants	480	890
Total	2670	4150

Remarks

- Autosubst 2 
- Curry-style simpler than Church-style
- First-Order using Equations tool

Website

<http://www.ps.uni-saarland.de/~spies/bachelor.php>

SU ⊑ **U**

$$\mathbf{SU}(E) \quad \text{iff} \quad \mathbf{U}(f(E))$$

Proof.

Pick $f := \{\Gamma \vdash s_i \stackrel{?}{=} t_i : A_i \mid i = 1, \dots, n\} \mapsto$

$$\Gamma \vdash \lambda h.h \ s_1 \cdots s_n \stackrel{?}{=} \lambda h.h \ t_1 \cdots t_n : A$$

where $A = (A_1 \rightarrow \cdots \rightarrow A_n \rightarrow \alpha) \rightarrow \alpha$. Follows with:

$$h \ u_1 \cdots u_n \equiv h \ v_1 \cdots v_n \quad \text{iff} \quad \forall i. \ u_i \equiv v_i$$

First-Order Unification

Traditionally

$$s, t ::= x \mid c \mid s \ t$$

This Work For normal forms:

$$\mathbf{U}_1(\Gamma \vdash_1 \lambda x_1 \cdots x_n.s \stackrel{?}{=} \lambda y_1 \cdots y_m.t : A)$$



$\mathbf{U}(s \stackrel{?}{=} t)$, without affecting bound variables

and $n = m$

First-Order Unification Algorithm

Unification

$$E \mapsto \sigma$$

$$\frac{\text{decomp } E = \text{nil}}{E \mapsto id}$$

$$\frac{\text{decomp } E = x \stackrel{?}{=} s :: E' \quad E'[s/x] \mapsto \sigma \\ \forall y \in \text{vars } s. \text{ free } y \quad \text{free } x \quad x \notin \text{vars } s}{E \mapsto \sigma[x := s[\sigma]]}$$

Example

	$\underline{g \ a \ b \stackrel{?}{=} g \ a \ b}$	$\underbrace{g \ x \ y \stackrel{?}{=} g \ (g \ y \ a) \ (g \ a \ a)}$	
		↓ decomp	
	$x \stackrel{?}{=} g \ y \ a$	$y \stackrel{?}{=} g \ a \ a$	

Conservativity — $\mathbf{U}_n \subseteq \mathbf{U}$

Let $\Gamma \vdash_n s \stackrel{?}{=} t : A$.

$s[\sigma] \equiv t[\sigma]$ for some $\Sigma \vdash_n \sigma : \Gamma$
iff

$s[\sigma] \equiv t[\sigma]$ for some $\Delta \vdash \sigma : \Gamma$

Proof Sketch.

Replace free variables and constants not of order n with first-order terms. For example, $x : (\alpha \rightarrow \alpha) \rightarrow \alpha$ is replaced by $\lambda x_1.z$ where $z : \alpha$ and $g : \alpha \rightarrow \alpha \rightarrow \alpha$ is replaced by $\lambda x_1x_2.z$. Normalise the result. □

Adding Constants

$$\mathbf{U}_n^{\mathcal{C}} \preceq \mathbf{U}_n^{\mathcal{D}} \quad \text{if } \mathcal{C} \subseteq \mathcal{D}$$

Proof Sketch.

Replace constants $d \in \mathcal{D} - \mathcal{C}$ with first-order terms, see conservativity. □

Removing Constans

$$\mathbf{U}_n^{\mathcal{D}} \preceq \mathbf{U}_n^{\mathcal{C}} \quad \text{if } \mathcal{C} \subseteq \mathcal{D} \text{ and } \forall d \notin C. \text{ord}(\Omega d) < n$$

Proof Sketch.

Let $\mathcal{C} = \{g\}$ and $\mathcal{D} = \{a, g\}$.

$$\begin{array}{ccc} g \ x \stackrel{?}{=} g \ a & \rightsquigarrow & \lambda x_a.g \ (x \ x_a) \stackrel{?}{=} \lambda x_a.g \ x_a \\ \text{where } x : \alpha & & \text{where } x : \alpha \rightarrow \alpha \end{array}$$

