Formalising the Undecidability of Higher-Order Unification

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2nd Bachelor Seminar Talk

Timeline

Timeline

G. Huet (1973)

PCP $\leq$ **U**$_3$

Timeline

G. Huet (1973)

$\mathbf{PCP} \leq \mathbf{U}_3$

W. Goldfarb (1981)

$\mathbf{H}_{10} \leq \mathbf{U}_2$

Timeline

Coq

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Example

$$\lambda xy.\ f x \stackrel{?}{=} \lambda xy.\ f y$$

Example

$$\lambda xy.\ f x \stackrel{?}{=} \lambda xy.\ f y$$

Example

$$\lambda xy.\ f\,x \stackrel{?}{=} \lambda xy.\ f\,y$$

Solution

$$\theta f = \lambda \_.\ z$$

$$\theta x = x \qquad \text{othw.}$$

Example

$$\lambda xy.\ f\,x \stackrel{?}{=} \lambda xy.\ f\,y$$

Solution

$$\theta f = \lambda\_.\ z$$

$$\theta x = x \qquad \text{othw.}$$

Proof

$$\begin{aligned}\lambda xy.\ (\theta f)\ x &\equiv \lambda xy.\ z\\ &\equiv \lambda xy.\ (\theta f)\ y\end{aligned}$$

Unification — **U**

$$\textbf{U}\ (\Gamma \vdash s \stackrel{?}{=} t : A)$$

$$\boxed{\Gamma \vdash s \stackrel{?}{=} t : A}$$

$$\frac{\Gamma \vdash s : A \quad \Gamma \vdash t : A}{\Gamma \vdash s \stackrel{?}{=} t : A}$$

Unification — U

$$\mathbf{U}\ (\Gamma \vdash s \overset{?}{=} t : A) \coloneqq$$
$$\exists \theta \qquad s[\theta] \ \equiv \ t[\theta]$$

$$\boxed{\Gamma \vdash s \overset{?}{=} t : A}$$

$$\boxed{s \ \equiv \ t}$$

$$\frac{\Gamma \vdash s : A \quad \Gamma \vdash t : A}{\Gamma \vdash s \overset{?}{=} t : A}$$

$$\frac{s \triangleright v \quad t \triangleright v}{s \ \equiv \ t}$$

Unification — U

$$\mathbf{U}\ (\Gamma \vdash s \stackrel{?}{=} t : A) \coloneqq$$
$$\exists \theta \Delta.\ \Delta \vdash \theta : \Gamma \quad \mathit{and} \quad s[\theta] \ \equiv\ t[\theta]$$

$$\boxed{\Gamma \vdash s \stackrel{?}{=} t : A} \qquad \boxed{\Delta \vdash \theta : \Gamma} \qquad \boxed{s \ \equiv\ t}$$

$$\frac{\Gamma \vdash s : A \quad \Gamma \vdash t : A}{\Gamma \vdash s \stackrel{?}{=} t : A} \qquad \frac{\forall (x : A) \in \Gamma.\ \Delta \vdash \theta x : A}{\Delta \vdash \theta : \Gamma} \qquad \frac{s \rhd v \quad t \rhd v}{s \ \equiv\ t}$$

nth-order Unification — $\mathbf{U}_n$

nth-order Unification — $\mathbf{U}_n$

$$\boxed{\Gamma \vdash s : A}$$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma \vdash s : A \to B \quad \Gamma \vdash t : A}{\Gamma \vdash s\ t : B}$$

$$\frac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \lambda x.\ s : A \to B}$$

nth-order Unification — $\mathbf{U}_n$

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$$\frac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \lambda x.\ s : A \to B} \qquad \frac{}{\Gamma \vdash c : \mathcal{C}\ c}$$

for signature $\mathcal{C}$: Const $\to$ Type

nth-order Unification — $\mathbf{U}_n$

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for signature $\mathcal{C}$: Const $\to$ Type

$$\boxed{\text{ord } A}$$

$$\text{ord } \alpha = 1 \qquad \text{ord } (A \to B) = \max\{\text{ord } A + 1,\ \text{ord } B\}$$

nth-order Unification — $\mathbf{U}_n$

$$\boxed{\Gamma \vdash_n s : A}$$

$$\frac{(x : A) \in \Gamma \quad \text{ord } A \leq n}{\Gamma \vdash_n x : A} \qquad \frac{\Gamma \vdash_n s : A \to B \quad \Gamma \vdash_n t : A}{\Gamma \vdash_n s\ t : B}$$

$$\frac{\Gamma, x : A \vdash_n s : B \quad \text{ord } A < n}{\Gamma \vdash_n \lambda x.\ s : A \to B} \qquad \frac{\text{ord } (\mathcal{C}\ c) \leq n}{\Gamma \vdash_n c : \mathcal{C}\ c}$$

for signature $\mathcal{C}$: Const $\to$ Type

$$\boxed{\text{ord } A}$$

$$\text{ord } \alpha = 1 \qquad \text{ord } (A \to B) = \max\{\text{ord } A + 1,\ \text{ord } B\}$$

Timeline

G. Huet (1973)

$\mathbf{PCP} \leq \mathbf{U}_3$

W. Goldfarb (1981)

$\mathbf{H}_{10} \leq \mathbf{U}_2$

Huet — Reduction

$$\mathbf{PCP}(C) \quad \textit{iff} \quad \mathbf{U}_3(f(C))$$

Huet — Reduction Function

$$f\,[\,x_1/y_1, \ldots, x_n/y_n\,] :=$$

Huet — Reduction Function

$$f\,[\,x_1/y_1,\ldots,x_n/y_n\,] :=$$

$$\lambda uvh.\, h\ (x_f\ \overline{x_1}\cdots\overline{x_n})\ (x_f\ u\cdots u) \stackrel{?}{=} \lambda uvh.\, h\ (x_f\ \overline{y_1}\cdots\overline{y_n})\ (u\ (d\ u))$$

Huet — Reduction Function

$$f\,[\,x_1/y_1, \ldots, x_n/y_n\,] :=$$

$$\lambda uvh.\, h\ (x_f\ \overline{x_1} \cdots \overline{x_n})\ (x_f\ u \cdots u) \stackrel{?}{=} \lambda uvh.\, h\ (x_f\ \overline{y_1} \cdots \overline{y_n})\ (u\ (d\ u))$$

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Huet — Reduction Function

$$\lambda uvh.\ h\ (x_f\ \overline{y_1} \cdots \overline{y_n})\ (u\ (d\ u))$$

Huet — List Calculus I

$$\boxed{s\ T}$$

$$s\ \mathsf{nil} = s$$

$$s\ (t :: T) = (s\ T)\ t$$

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$$\boxed{S\ t}$$

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$$(s :: S)\ t = s\ (S\ t)$$

Huet — List Calculus I

$$\boxed{s\ T}$$

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$$\boxed{S\ t}$$

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$$(s :: S)\ t = s\ (S\ t)$$

$$\boxed{\Lambda_X.s}$$

$$\Lambda_{\mathsf{nil}}.s = s$$

$$\Lambda_{x::X}.s = \lambda x.\ \Lambda_X.s$$

Huet — List Calculus I

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$$(s :: S)\ t = s\ (S\ t)$$

$$\boxed{\Lambda_X.s}$$

$$\Lambda_{\mathsf{nil}}.s = s$$

$$\Lambda_{x::X}.s = \lambda x.\ \Lambda_X.s$$

$$\boxed{L \to A}$$

$$\mathsf{nil} \to A = A$$

$$(B :: L) \to A = B \to (L \to A)$$

Huet — List Calculus II

$$\boxed{\Gamma \vdash S : L}$$

$$\frac{}{\Gamma \vdash \mathsf{nil} : \mathsf{nil}}$$

$$\frac{\Gamma \vdash s : A \quad \Gamma \vdash S : L}{\Gamma \vdash s :: S : A :: L}$$

Huet — List Calculus II

$$\boxed{\Gamma \vdash S : L}$$

$$\frac{}{\Gamma \vdash \mathsf{nil} : \mathsf{nil}}$$

$$\frac{\Gamma \vdash s : A \quad \Gamma \vdash S : L}{\Gamma \vdash s :: S : A :: L}$$

$$\boxed{S \succ S'}$$

$$\frac{s \succ s'}{s :: S \succ s' :: S}$$

$$\frac{S \succ S'}{s :: S \succ s :: S'}$$

Huet — Busy Work

$$s\ (T_1 +\!\!+\ T_2) = (s\ T_2)\ T_1 \qquad (s\ T)[\xi] = s[\xi]\ T[\xi]$$

$$(s\ T)[\theta] = s[\theta]\ T[\theta] \qquad \frac{T \succ T'}{s\ T \succ s\ T'} \qquad \frac{s \succ s'}{s\ T \succ s'\ T}$$

$$\frac{s \succ^* s' \quad T \succ^* T'}{s\ T \succ^* s'\ T'} \qquad \frac{s \equiv s' \quad T \equiv T'}{s\ T \equiv s'\ T'}$$

$$\frac{\Gamma \vdash s : \text{rev}\ L \to A \quad \Gamma \vdash T : L}{\Gamma \vdash s\ T : A}$$

- if normal $(s\ T)$ then normal s and normal T
- if $\Gamma \vdash s\ T : B$ then $\Gamma \vdash T : L$, $\Gamma \vdash s : \text{rev}\ L \to A$ for some L

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H$_{10}$ $\leq$ **SU**$_2$ $\leq$ **U**$_2$

Goldfarb — Diophantine Equations

$$e ::= \;\; x + y = z \;\;|\;\; x \cdot y = z \;\;|\;\; x = c \quad (c \in \mathbb{N})$$

Goldfarb — Diophantine Equations

$$e ::= \; x + y = z \;\mid\; x \cdot y = z \;\mid\; x = c \quad (c \in \mathbb{N})$$

$$\boxed{\sigma \models e}$$

$$\begin{array}{lll} \sigma \models x = c & \textit{iff} & \sigma x = c \\ \sigma \models x + y = z & \textit{iff} & \sigma x + \sigma y = \sigma z \\ \sigma \models x \cdot y = z & \textit{iff} & \sigma x \cdot \sigma y = \sigma z \end{array}$$

Goldfarb — Diophantine Equations

$$e ::= \ x + y = z \ \mid \ x \cdot y = z \ \mid \ x = c \quad (c \in \mathbb{N})$$

$\boxed{\sigma \vDash e}$

$$\sigma \vDash x = c \quad \textit{iff} \quad \sigma x = c$$
$$\sigma \vDash x + y = z \quad \textit{iff} \quad \sigma x + \sigma y = \sigma z$$
$$\sigma \vDash x \cdot y = z \quad \textit{iff} \quad \sigma x \cdot \sigma y = \sigma z$$

$\boxed{\sigma \vDash E}$

$$\frac{\forall e \in E.\ \sigma \vDash e}{\sigma \vDash E}$$

Goldfarb — Diophantine Equations

$$\mathbf{H}_{10}(E) := \exists \sigma.\ \sigma \vDash E$$

$$e ::= \ x + y = z \ \mid \ x \cdot y = z \ \mid \ x = c \quad (c \in \mathbb{N})$$

$\boxed{\sigma \vDash e}$

$$\sigma \vDash x = c \quad \textit{iff} \quad \sigma x = c$$
$$\sigma \vDash x + y = z \quad \textit{iff} \quad \sigma x + \sigma y = \sigma z$$
$$\sigma \vDash x \cdot y = z \quad \textit{iff} \quad \sigma x \cdot \sigma y = \sigma z$$

$\boxed{\sigma \vDash E}$

$$\frac{\forall e \in E.\ \sigma \vDash e}{\sigma \vDash E}$$

Goldfarb — Reduction

$$\mathbf{H}_{10}(E) \quad \textit{iff} \quad \mathbf{SU}_2(F(E))$$

Goldfarb — Reduction Function

Constants

$$c ::= \mathsf{a} \mid \mathsf{b} \mid \mathsf{g}$$

Goldfarb — Reduction Function

Constants

$$c ::= \mathsf{a} \mid \mathsf{b} \mid \mathsf{g}$$

$$\mathcal{C}\ \mathsf{a} = \alpha \qquad \mathcal{C}\ \mathsf{b} = \alpha$$

$$\mathcal{C}\ \mathsf{g} = \alpha \to \alpha \to \alpha$$

Goldfarb — Reduction Function

Constants

$$c ::= \mathsf{a} \mid \mathsf{b} \mid \mathsf{g}$$

$$\mathcal{C}\ \mathsf{a} = \alpha \qquad \mathcal{C}\ \mathsf{b} = \alpha$$

$$\mathcal{C}\ \mathsf{g} = \alpha \to \alpha \to \alpha$$

$$\boxed{\overline{n}\ t}$$

$$\overline{0}\ t = t$$

$$\overline{\mathsf{S}n}\ t = \mathsf{g}\ \mathsf{a}\ (\overline{n}\ t)$$

Goldfarb — Reduction Function

Constants

$$c ::= \mathsf{a} \mid \mathsf{b} \mid \mathsf{g}$$

$$\mathcal{C}\ \mathsf{a} = \alpha \qquad \mathcal{C}\ \mathsf{b} = \alpha$$

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Example $(x = 42) \mapsto$

Goldfarb — Reduction Function

Constants

$$c ::= \mathsf{a} \mid \mathsf{b} \mid \mathsf{g}$$

$$\mathcal{C}\ \mathsf{a} = \alpha \qquad \mathcal{C}\ \mathsf{b} = \alpha$$

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$$\boxed{\overline{n}\ t}$$

$$\overline{0}\ t = t$$

$$\overline{\mathsf{S}n}\ t = \mathsf{g}\ \mathsf{a}\ (\overline{n}\ t)$$

Example $(x = 42) \mapsto$

$$x\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (x\ \mathsf{a})$$

Goldfarb — Reduction Function

Constants

$$c ::= \mathsf{a} \mid \mathsf{b} \mid \mathsf{g}$$

$$\mathcal{C}\ \mathsf{a} = \alpha \qquad \mathcal{C}\ \mathsf{b} = \alpha$$

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$$\boxed{\overline{n}\ t}$$

$$\overline{0}\ t = t$$

$$\overline{\mathsf{S}n}\ t = \mathsf{g}\ \mathsf{a}\ (\overline{n}\ t)$$

Example $(x = 42) \mapsto$

$$x\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (x\ \mathsf{a}) \qquad x\ \mathsf{a} \stackrel{?}{=} \overline{42}\ \mathsf{a}$$

Goldfarb — Multiplication

$$(x \cdot y = z) \mapsto$$

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$$x\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (x\ \mathsf{a}) \qquad y\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (y\ \mathsf{a}) \qquad z\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (z\ \mathsf{a})$$

Goldfarb — Multiplication

$$(x \cdot y = z) \mapsto$$

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$$G\ \mathsf{a}\ \mathsf{b}\ (\mathsf{g}\ (\mathsf{g}\ (z\ \mathsf{a})(y\ \mathsf{b}))\ \mathsf{a}) \stackrel{?}{=} \mathsf{g}\ (\mathsf{g}\ \mathsf{a}\ \mathsf{b})(G\ (x\ \mathsf{a})\ (\overline{1}\ \mathsf{b})\ \mathsf{a})$$

$$G\ \mathsf{b}\ \mathsf{a}\ (\mathsf{g}\ (\mathsf{g}\ (z\ \mathsf{b})(y\ \mathsf{a}))\ \mathsf{a}) \stackrel{?}{=} \mathsf{g}\ (\mathsf{g}\ \mathsf{b}\ \mathsf{a})(G\ (x\ \mathsf{b})\ (\overline{1}\ \mathsf{a})\ \mathsf{a})$$

Goldfarb — Multiplication

$$(x \cdot y = z) \mapsto$$

$$x\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (x\ \mathsf{a}) \qquad y\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (y\ \mathsf{a}) \qquad z\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (z\ \mathsf{a})$$

$$G\ \mathsf{a}\ \mathsf{b}\ (\mathsf{g}\ (\mathsf{g}\ (z\ \mathsf{a})(y\ \mathsf{b}))\ \mathsf{a}) \stackrel{?}{=} \mathsf{g}\ (\mathsf{g}\ \mathsf{a}\ \mathsf{b})(G\ (x\ \mathsf{a})\ (\overline{1}\ \mathsf{b})\ \mathsf{a})$$

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Goldfarb — Multiplication

$$(x \cdot y = z) \mapsto$$

$$x\ (\overline{1}\ \mathsf{a}) \overset{?}{=} \overline{1}\ (x\ \mathsf{a}) \qquad y\ (\overline{1}\ \mathsf{a}) \overset{?}{=} \overline{1}\ (y\ \mathsf{a}) \qquad z\ (\overline{1}\ \mathsf{a}) \overset{?}{=} \overline{1}\ (z\ \mathsf{a})$$

$$G_{xyz}\ \mathsf{a}\ \mathsf{b}\ (\mathsf{g}\ (\mathsf{g}\ (z\ \mathsf{a})(y\ \mathsf{b}))\ \mathsf{a}) \overset{?}{=} \mathsf{g}\ (\mathsf{g}\ \mathsf{a}\ \mathsf{b})(G_{xyz}\ (x\ \mathsf{a})\ (\overline{1}\ \mathsf{b})\ \mathsf{a})$$

$$G_{xyz}\ \mathsf{b}\ \mathsf{a}\ (\mathsf{g}\ (\mathsf{g}\ (z\ \mathsf{b})(y\ \mathsf{a}))\ \mathsf{a}) \overset{?}{=} \mathsf{g}\ (\mathsf{g}\ \mathsf{b}\ \mathsf{a})(G_{xyz}\ (x\ \mathsf{b})\ (\overline{1}\ \mathsf{a})\ \mathsf{a})$$

Goldfarb — Assignment $\sigma \to$ Substitution θ

Substitution

$$\theta u = \begin{cases} \end{cases}$$

Goldfarb — Assignment $\sigma \to$ Substitution θ

Substitution

$$\theta u = \begin{cases} \lambda z.\ \overline{\sigma u}\ z & u \in \mathsf{Vars}\ E \\ \end{cases}$$

Goldfarb — Assignment $\sigma \to$ Substitution θ

Substitution

$$\theta u = \begin{cases} \lambda z.\ \overline{\sigma u}\ z & u \in \mathsf{Vars}\ E \\ \lambda w_1 w_2 w_3.\ \mathsf{g}\ t_0^{\sigma y} (\cdots (\mathsf{g}\ t_{\sigma x - 1}^{\sigma y}\ w_3)) & u = G_{xyz} \end{cases}$$

where $t_k^m := \mathsf{g}\ (\overline{k \cdot m}\ w_1)\ (\overline{k}\ w_2)$

Goldfarb — Substitution $\theta \to$ Assignment σ

$(x = 1) \mapsto$

$$x\ \mathsf{a} \stackrel{?}{=} \overline{1}\ \mathsf{a} \quad \textit{and} \quad x\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (x\ \mathsf{a})$$

How many substitutions θ?

Goldfarb — Substitution $\theta \to$ Assignment σ

$(x = 1) \mapsto$

$$x\ \mathsf{a} \stackrel{?}{=} \overline{1}\ \mathsf{a} \quad \textit{and} \quad x\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (x\ \mathsf{a})$$

How many substitutions θ?

$\theta x = \lambda x.\ \overline{1}\ x$

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How many substitutions θ?

$\theta x = \lambda x. \ \overline{1} \ x \qquad \theta x = \mathsf{g} \ \mathsf{a}$

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$(x = 1) \mapsto$

$$x\ \mathsf{a} \overset{?}{\doteq} \overline{1}\ \mathsf{a} \quad \textit{and} \quad x\ (\overline{1}\ \mathsf{a}) \overset{?}{\doteq} \overline{1}\ (x\ \mathsf{a})$$

How many substitutions θ?

$\theta x = \lambda x.\ \overline{1}\ x \qquad \theta x = \mathsf{g}\ \mathsf{a} \qquad \theta x = \lambda x.\ (\lambda y.\ y)\ (\overline{1}\ x)$

Goldfarb — Substitution $\theta \to$ Assignment σ

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How many substitutions θ?

$$\theta x = \lambda x.\ \overline{1}\ x \qquad \theta x = \mathsf{g}\ \mathsf{a} \qquad \theta x = \lambda x.\ (\lambda y.\ y)\ (\overline{1}\ x) \qquad \cdots$$

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$(x = 1) \mapsto$

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How many substitutions θ?

$$\theta x = s \quad \text{where} \quad s\ \mathsf{a} \equiv \overline{1}\ \mathsf{a}$$

Goldfarb — Substitution $\theta \to$ Assignment σ

$(x = 1) \mapsto$

$$x \; \mathsf{a} \stackrel{?}{=} \overline{1} \; \mathsf{a} \quad \textit{and} \quad x \; (\overline{1} \; \mathsf{a}) \stackrel{?}{=} \overline{1} \; (x \; \mathsf{a})$$

Algorithm σx

1. Decide $\Sigma n.\ \theta x \; \mathsf{a} \equiv \overline{n} \; \mathsf{a}$

Goldfarb — Substitution $\theta \to$ Assignment σ

$(x = 1) \mapsto$

$$x\ \mathsf{a} \stackrel{?}{=} \overline{1}\ \mathsf{a} \quad \mathit{and} \quad x\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (x\ \mathsf{a})$$

Algorithm σx

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Algorithm σx

1. Use $\Delta \vdash \theta : \Gamma$ to normalise θx — $\exists s.\ \theta x \rhd s$

Goldfarb — Substitution $\theta \to$ Assignment σ

$(x = 1) \mapsto$

$$x \ \mathsf{a} \stackrel{?}{=} \overline{1} \ \mathsf{a} \quad \textit{and} \quad x \ (\overline{1} \ \mathsf{a}) \stackrel{?}{=} \overline{1} \ (x \ \mathsf{a})$$

Algorithm σx

1. Use $\Delta \vdash \theta : \Gamma$ to normalise θx — $\exists s.\ \theta x \triangleright s$
2. Circumvent elim restriction — $\Sigma s.\ \theta x \triangleright s$

Goldfarb — Substitution $\theta \to$ Assignment σ

$(x = 1) \mapsto$

$$x\ \mathsf{a} \stackrel{?}{=} \overline{1}\ \mathsf{a} \quad \mathit{and} \quad x\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (x\ \mathsf{a})$$

Algorithm σx

1. Use $\Delta \vdash \theta : \Gamma$ to normalise θx — $\exists s.\ \theta x \rhd s$
2. Circumvent elim restriction — $\Sigma s.\ \theta x \rhd s$
3. Normalise $(s\ \mathsf{a})$ — $s\ \mathsf{a} \rhd t$

Goldfarb — Substitution $\theta \to$ Assignment σ

$(x = 1) \mapsto$

$$x\ \mathsf{a} \stackrel{?}{=} \overline{1}\ \mathsf{a} \quad \textit{and} \quad x\ (\overline{1}\ \mathsf{a}) \stackrel{?}{=} \overline{1}\ (x\ \mathsf{a})$$

Algorithm σx

1. Use $\Delta \vdash \theta : \Gamma$ to normalise θx — $\exists s.\ \theta x \triangleright s$
2. Circumvent elim restriction — $\Sigma s.\ \theta x \triangleright s$
3. Normalise $(s\ \mathsf{a})$ — $s\ \mathsf{a} \triangleright t$
4. Decide $\Sigma n.\ t = \overline{n}\ \mathsf{a}$

Conclusion

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G. Huet (1973)

$$\mathbf{PCP} \leq \mathbf{U}_3$$

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Conclusion

Contributions

- Order Typing $\vdash_n$
- Unified Calculus

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Contributions

- Order Typing $\vdash_n$
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Future Work

- Monadic 2nd-order Unification
- Semi-Unification
- System F typeability
- System F inhabitation

Related Work I

G. Huet.
The undecidability of unification in third order logic.
Information and control, 1973.

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Formalisation

Contents	**Spec**	**Proofs**
Preliminaries	250	300
λ-Calculus	300	340
Confluence	110	180
Weak Normalisation	60	100
List Calculus	200	340
Huet	140	420
Goldfarb	380	1000
Total	1440	2680

Set Unification

$$\mathbf{SU}_n \{\Gamma \vdash_n s_i \stackrel{?}{=} t_i : \alpha \mid 1 \leq i \leq k\} :=$$
$$\exists \theta \Delta.\ \Delta \vdash \theta : \Gamma \quad and \quad \forall i.\ s_i[\theta] \ \equiv \ t_i[\theta]$$

$\mathbf{SU}_2 \ \leq \ \mathbf{U}_2$

$$\{\Gamma \vdash_2 s_i \stackrel{?}{=} t_i : \alpha \mid 1 \leq i \leq k\} \mapsto$$

$$\Gamma \vdash_2 \mathsf{g}\ s_1\ (\cdots\ (\mathsf{g}\ s_k\ \mathsf{a})) \stackrel{?}{=} \mathsf{g}\ t_1\ (\cdots\ (\mathsf{g}\ t_k\ \mathsf{a})) : \alpha$$

Lemma

$\mathsf{g}\ s_1\ s_2 \equiv \mathsf{g}\ t_1\ t_2$ *iff* $\forall i.\ s_i \equiv t_i$

Goldfarb Reduction — $x \cdot y = z$

$$(G_{xyz}\ \mathsf{a}\ \mathsf{b}\ (\mathsf{g}\ (\mathsf{g}\ (z\ \mathsf{a})(y\ \mathsf{b}))\ \mathsf{a})) \stackrel{?}{=} \mathsf{g}\ (\mathsf{g}\ \mathsf{a}\ \mathsf{b})(G_{xyz}\ (x\ \mathsf{a})\ (\overline{1}\ \mathsf{b})\ \mathsf{a})$$

$$(G_{xyz}\ \mathsf{b}\ \mathsf{a}\ (\mathsf{g}\ (\mathsf{g}\ (z\ \mathsf{b})(y\ \mathsf{a}))\ \mathsf{a})) \stackrel{?}{=} \mathsf{g}\ (\mathsf{g}\ \mathsf{b}\ \mathsf{a})(G_{xyz}\ (x\ \mathsf{b})\ (\overline{1}\ \mathsf{a})\ \mathsf{a})$$

Goldfarb Reduction — $x \cdot y = z$

$$(G_{xyz}\ \mathsf{a}\ \mathsf{b}\ (\mathsf{g}\ (\mathsf{g}\ (z\ \mathsf{a})(y\ \mathsf{b}))\ \mathsf{a}))[\theta] \equiv \mathsf{g}\ (\mathsf{g}\ \mathsf{a}\ \mathsf{b})(G_{xyz}\ (x\ \mathsf{a})\ (\overline{1}\ \mathsf{b})\ \mathsf{a})[\theta]$$
$$(G_{xyz}\ \mathsf{b}\ \mathsf{a}\ (\mathsf{g}\ (\mathsf{g}\ (z\ \mathsf{b})(y\ \mathsf{a}))\ \mathsf{a}))[\theta] \equiv \mathsf{g}\ (\mathsf{g}\ \mathsf{b}\ \mathsf{a})(G_{xyz}\ (x\ \mathsf{b})\ (\overline{1}\ \mathsf{a})\ \mathsf{a})[\theta]$$

where

$$\theta x\ t \equiv \overline{m}\ t \qquad \theta y\ t \equiv \overline{n}\ t \qquad \theta z\ t \equiv \overline{p}\ t$$

Goldfarb Reduction — $x \cdot y = z$

$$(G_{xyz}[\theta] \text{ a b } (\text{g } (\text{g } (\overline{p} \text{ a})(\overline{n} \text{ b})) \text{ a})) \equiv \text{g } (\text{g a b})(G_{xyz}[\theta] \ (\overline{m} \text{ a}) \ (\overline{1} \text{ b}) \text{ a})$$

$$(G_{xyz}[\theta] \text{ b a } (\text{g } (\text{g } (\overline{p} \text{ b})(\overline{n} \text{ a})) \text{ a})) \equiv \text{g } (\text{g b a})(G_{xyz}[\theta] \ (\overline{m} \text{ b}) \ (\overline{1} \text{ a}) \text{ a})$$

where

$$\theta x \ t \equiv \overline{m} \ t \qquad \theta y \ t \equiv \overline{n} \ t \qquad \theta z \ t \equiv \overline{p} \ t$$

Goldfarb Reduction — $x \cdot y = z$

By analysis of θG_{xyz} we have $\theta G_{xyz} \triangleright \lambda w_1 w_2 w_3.\ u$

$$u[\sigma_1] \equiv \mathsf{g}\ (\mathsf{g}\ \mathsf{a}\ \mathsf{b})\ (u[\tau_1])$$
$$u[\sigma_2] \equiv \mathsf{g}\ (\mathsf{g}\ \mathsf{b}\ \mathsf{a})\ (u[\tau_2])$$

	w_1	w_2	w_3
σ_1	a	b	g (g ($\overline{p}$ a) ($\overline{n}$ b)) a
σ_2	b	a	g (g ($\overline{p}$ b) ($\overline{n}$ a)) a
τ_1	$\overline{m}$ a	$\overline{1}$ b	a
τ_2	$\overline{m}$ b	$\overline{1}$ a	a

Goldfarb Redution — $x \cdot y = z$

$$u[\sigma_1] \equiv \mathsf{g}\ \ (\mathsf{g}\ \ \mathsf{a}\ \ \mathsf{b})\ \ u[\tau_1]$$
$$u[\sigma_2] \equiv \mathsf{g}\ \ (\mathsf{g}\ \ \mathsf{b}\ \ \mathsf{a})\ \ u[\tau_2]$$

Substitutions

	w_1	w_2	w_3
σ_1	a	b	g (g ($\overline{p}$ a) ($\overline{n}$ b)) a
σ_2	b	a	g (g ($\overline{p}$ b) ($\overline{n}$ a)) a
τ_1	$\overline{m}$ a	$\overline{1}$ b	a
τ_2	$\overline{m}$ b	$\overline{1}$ a	a

Goldfarb Redution — $x \cdot y = z$

$$u[\sigma_1] \equiv \mathsf{g}\ \ t_0[\sigma_1]\ \ u[\tau_1]$$
$$u[\sigma_2] \equiv \mathsf{g}\ \ t_0[\sigma_2]\ \ u[\tau_2]$$

where $t_k := \mathsf{g}\ (\overline{k \cdot m}\ w_1)\ (\overline{k}\ w_2)$

Substitutions

	w_1	w_2	w_3
σ_1	a	b	g (g ($\overline{p}$ a) ($\overline{n}$ b)) a
σ_2	b	a	g (g ($\overline{p}$ b) ($\overline{n}$ a)) a
τ_1	$\overline{m}$ a	$\overline{1}$ b	a
τ_2	$\overline{m}$ b	$\overline{1}$ a	a

Goldfarb Redution — $x \cdot y = z$

$$\mathsf{g}\ \ t_0[\sigma_1]\ \ u'[\sigma_1] \equiv \mathsf{g}\ \ t_0[\sigma_1]\ \ (\mathsf{g}\ \ t_0[\tau_1]\ \ u'[\tau_1])$$
$$\mathsf{g}\ \ t_0[\sigma_2]\ \ u'[\sigma_2] \equiv \mathsf{g}\ \ t_0[\sigma_2]\ \ (\mathsf{g}\ \ t_0[\tau_2]\ \ u'[\tau_2])$$

where $t_k := \mathsf{g}\ (\overline{k \cdot m}\ w_1)\ (\overline{k}\ w_2)$

Substitutions

	w_1	w_2	w_3
σ_1	a	b	g (g ($\overline{p}$ a) ($\overline{n}$ b)) a
σ_2	b	a	g (g ($\overline{p}$ b) ($\overline{n}$ a)) a
τ_1	$\overline{m}$ a	$\overline{1}$ b	a
τ_2	$\overline{m}$ b	$\overline{1}$ a	a

Goldfarb Redution — $x \cdot y = z$

$$\mathrm{g}\ \ t_0[\sigma_1]\ \ u'[\sigma_1] \equiv \mathrm{g}\ \ t_0[\sigma_1]\ \ (\mathrm{g}\ \ t_1[\sigma_1]\ \ u'[\tau_1])$$
$$\mathrm{g}\ \ t_0[\sigma_2]\ \ u'[\sigma_2] \equiv \mathrm{g}\ \ t_0[\sigma_2]\ \ (\mathrm{g}\ \ t_1[\sigma_2]\ \ u'[\tau_2])$$

where $t_k := \mathrm{g}\ (\overline{k \cdot m}\ w_1)\ (\overline{k}\ w_2)$

Substitutions

	w_1	w_2	w_3
σ_1	a	b	g (g ($\overline{p}$ a) ($\overline{n}$ b)) a
σ_2	b	a	g (g ($\overline{p}$ b) ($\overline{n}$ a)) a
τ_1	$\overline{m}$ a	$\overline{1}$ b	a
τ_2	$\overline{m}$ b	$\overline{1}$ a	a

Lemma

$$t_k[\tau_i] \equiv t_{k+1}[\sigma_i]$$

Goldfarb Redution — $x \cdot y = z$

$$u''[\sigma_1] \equiv \mathsf{g}\ \ t_n[\sigma_1]\ \ u''[\tau_1]$$
$$u''[\sigma_2] \equiv \mathsf{g}\ \ t_n[\sigma_2]\ \ u''[\tau_2]$$

where $t_k := \mathsf{g}\ (\overline{k \cdot m}\ w_1)\ (\overline{k}\ w_2)$

Substitutions

	w_1	w_2	w_3
σ_1	a	b	g (g ($\overline{p}$ a) ($\overline{n}$ b)) a
σ_2	b	a	g (g ($\overline{p}$ b) ($\overline{n}$ a)) a
τ_1	$\overline{m}$ a	$\overline{1}$ b	a
τ_2	$\overline{m}$ b	$\overline{1}$ a	a

Lemma

$$t_k[\tau_i] \equiv t_{k+1}[\sigma_i]$$

Goldfarb Redution — $x \cdot y = z$

$$u''[\sigma_1] \equiv \mathsf{g} \;\; (\mathsf{g}\ (\overline{n \cdot m}\ w_1)\ (\overline{n}\ w_2))[\sigma_1] \;\; u''[\tau_1]$$
$$u''[\sigma_2] \equiv \mathsf{g} \;\; (\mathsf{g}\ (\overline{n \cdot m}\ w_1)\ (\overline{n}\ w_2))[\sigma_2] \;\; u''[\tau_2]$$

where $t_k := \mathsf{g}\ (\overline{k \cdot m}\ w_1)\ (\overline{k}\ w_2)$

Substitutions

	w_1	w_2	w_3
σ_1	a	b	g (g ($\overline{p}$ a) ($\overline{n}$ b)) a
σ_2	b	a	g (g ($\overline{p}$ b) ($\overline{n}$ a)) a
τ_1	$\overline{m}$ a	$\overline{1}$ b	a
τ_2	$\overline{m}$ b	$\overline{1}$ a	a

Lemma

$$t_k[\tau_i] \equiv t_{k+1}[\sigma_i]$$

Goldfarb Redution — $x \cdot y = z$

$$w_3[\sigma_1] \equiv \mathsf{g} \;\; (\mathsf{g}\ (\overline{n \cdot m}\ w_1)\ (\overline{n}\ w_2))[\sigma_1] \quad w_3[\tau_1]$$
$$w_3[\sigma_2] \equiv \mathsf{g} \;\; (\mathsf{g}\ (\overline{n \cdot m}\ w_1)\ (\overline{n}\ w_2))[\sigma_2] \quad w_3[\tau_2]$$

where $t_k := \mathsf{g}\ (\overline{k \cdot m}\ w_1)\ (\overline{k}\ w_2)$

Substitutions

	w_1	w_2	w_3
σ_1	a	b	g (g ($\overline{p}$ a) ($\overline{n}$ b)) a
σ_2	b	a	g (g ($\overline{p}$ b) ($\overline{n}$ a)) a
τ_1	$\overline{m}$ a	$\overline{1}$ b	a
τ_2	$\overline{m}$ b	$\overline{1}$ a	a

Lemma

$$t_k[\tau_i] \equiv t_{k+1}[\sigma_i]$$