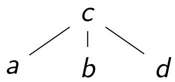
Formalizing Nu-Tree Automata

Leonhard Staut
Advisor: Dominik Kirst
Supervisor: Prof. Dr. Gert Smolka
Saarland University
June 30, 2017

Recap

- Tree languages over infinite alphabet $\mathbb{A}$ with similar structure arising from systematic permutation of names

$\{$  $\mid a, b \in \mathbb{A} \setminus \{d, c\}; a \neq b \}$

- Representation: $\boldsymbol{\nu}$-Trees

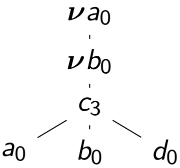
$\boldsymbol{\nu}$-Tree

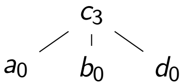
Definition ($\boldsymbol{\nu}$-Tree [Kirst, 2016])

The set $\boldsymbol{\nu}$-Tree is defined inductively by

$$n ::= a_k n_1 \dots n_k \mid \boldsymbol{\nu} a_k . n$$

where a_k ranges over the countably infinite ranked alphabet $\mathbb{A}$.

- A $\boldsymbol{\nu}$-tree n denotes a set of pure trees $[\![n]\!]$ with
  - Same structure
  - Instantiated $\boldsymbol{\nu}$ bindings with fresh names
- 

  $\rightsquigarrow$ $\{$ 

  $\mid a_0, b_0 \in \mathbb{A} \setminus \{d_0, c_3\}; a_0 \neq b_0 \}$

$\boldsymbol{\nu}$-Tree Denotation

Definition ($\boldsymbol{\nu}$-Tree Denotation)

$$\frac{t_i \in [\![n_i]\!]_{a_k::A}}{a_k t_1 \ldots t_k \in [\![a_k n_1 \ldots n_k]\!]_A}$$

$$\frac{t \in [\![(a_k b_k) \cdot n]\!]_{b_k::A} \qquad b_k \notin A \quad b_k \notin \mathtt{FN}(\boldsymbol{\nu} a_k.n)}{t \in [\![\boldsymbol{\nu} a_k.n]\!]_A}$$

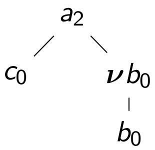
$$\pi \cdot (a_k n_1 \ldots n_k) = (\pi a_k)(\pi \cdot n_1) \ldots (\pi \cdot n_k)$$

$$\pi \cdot (\boldsymbol{\nu} a_k.n) = \boldsymbol{\nu}(\pi a_k).(\pi \cdot n)$$

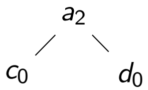
- Equivariance: $\pi \cdot [\![n]\!]_A = [\![\pi \cdot n]\!]_{\pi \cdot A}$
- Closed under permutations of bound names: π fixes $\mathtt{FN}(n) \rightarrow [\![\pi \cdot n]\!]_A = [\![n]\!]_A$

Decide $t \in [\![n]\!]_A$: Example

$n :=$



$t :=$



- Match up names and structure
- Correctly instantiate all name bindings

Decide $t \in [\![n]\!]_A$

$$\text{d\_dec } A\ (a_k n_1 \dots n_k)\ (a_k t_1 \dots t_k) := \bigwedge_{i=1,\dots,k} \text{d\_dec } (a_k :: A)\ n_i\ t_i$$

$$\text{d\_dec } A\ (\boldsymbol{\nu} a_k.n)\ t := \begin{cases} \displaystyle\bigvee_{\text{candidates } b_k} \text{d\_dec } (b_k :: A)\ ((a_k b_k) \cdot n)\ t & \text{, if } a_k \in \mathtt{FN}(n) \\ \text{d\_dec } A\ n\ t & \text{, if } a_k \notin \mathtt{FN}(n) \end{cases}$$

where candidates is the set of possible instantiations for a_k:

$$\text{candidates } a_k\ A\ n\ := \mathtt{Name}_k(t) \setminus (A \bigcup \mathtt{FN}(\boldsymbol{\nu} a_k.n))$$

Double induction for $\boldsymbol{\nu}$-Tree and $\mathbb{A}$-Tree

- Define inductive predicate on a $\boldsymbol{\nu}$-tree and a pure tree sharing a similar structure

Definition (Sim Induction)

$$\frac{n_i \approx t_i}{a_k n_1 \ldots n_k \approx b_k t_1 \ldots t_k} \qquad \frac{n \approx t}{\boldsymbol{\nu} a_k . n \approx t}$$

- Show
  - d_dec $A\ n\ t = \text{true} \rightarrow n \approx t$
  - $t \in [\![n]\!]_A \rightarrow n \approx t$

Correctness of d_dec

Theorem

$t \in [\![n]\!]_A \leftrightarrow \text{d\_dec}\ A\ n\ t = \text{true}$

- Proof by induction on $n \approx t$

Lemma

- $\forall A\ n\ t\ a_k.\ a_k \in \texttt{FN}(n) \to t \in [\![n]\!]_A \to a_k \in \texttt{Name}(t)$
- $\forall A\ n\ t\ a_k.\ a_k \notin \texttt{FN}(n) \to t \in [\![n]\!]_A \leftrightarrow t \in [\![\boldsymbol{\nu} a_k.n]\!]_A$

Automaton model for ν-trees

Definition (NTA [Stirling, 2009])

An NTA $\mathcal{A}$ is a triple $(\mathbb{A}, Q, \Delta)$

- Countably infinite ranked alphabet $\mathbb{A}$
- Finite set of states Q
- Finite set of transitions Δ

$$\Delta_1 \subseteq \{q \stackrel{a_k}{\Rightarrow} [q_1, \ldots, q_k] \mid q, q_1, \ldots, q_k \in Q, a_k \in \mathbb{A}\}$$

$$\Delta_2 \subseteq \{(q, q') \stackrel{a_k}{\Rightarrow} [q_1, \ldots, q_k] \mid q, q', q_1, \ldots, q_k \in Q, a_k \in \mathbb{A}\}$$

$$\Delta_3 \subseteq \{q \stackrel{a_k}{\Rightarrow} q' \mid q, q' \in Q, a_k \in \mathbb{A}\}$$

NTA language

Definition (NTA language)

The language of an NTA $\mathcal{L}(\mathcal{A}, q, \varphi)$ is defined relative to $q \in Q$ and $\varphi : \mathbb{A} \to Q_\perp$:

$$\frac{\varphi(a_k) = \perp \quad (q \overset{a_k}{\Rightarrow} [q_1, \ldots, q_k]) \in \Delta_1 \quad n_i \in \mathcal{L}(\mathcal{A}, q_i, \varphi)}{a_k n_1 \ldots n_k \in \mathcal{L}(\mathcal{A}, q, \varphi)}$$

$$\frac{\varphi(a_k) = q' \quad ((q, q') \overset{a_k}{\Rightarrow} [q_1, \ldots, q_k]) \in \Delta_2 \quad n_i \in \mathcal{L}(\mathcal{A}, q_i, \varphi)}{a_k n_1 \ldots n_k \in \mathcal{L}(\mathcal{A}, q, \varphi)}$$

$$\frac{(q \overset{a_k}{\Rightarrow} q') \in \Delta_3 \quad n \in \mathcal{L}(\mathcal{A}, q', \varphi[a_k := q])}{\nu a_k.n \in \mathcal{L}(\mathcal{A}, q, \varphi)}$$

Decide $n \in \mathcal{L}(\mathcal{A}, q, \varphi)$

For given q and φ:

$$\text{a\_dec } \mathcal{A} \;\; (a_k n_1 \ldots n_k) \; q \; \varphi \quad := \bigvee_{(q \overset{a_k}{\Rightarrow} [q_1,\ldots,q_k]) \in \Delta_1} \Big(\bigwedge_{i=1\ldots k} (\text{a\_dec } \mathcal{A} \; n_i \; q_i \; \varphi)\Big) \text{ , if } \varphi(a_k) = \bot$$

$$\text{a\_dec } \mathcal{A} \;\; (a_k n_1 \ldots n_k) \; q \; \varphi \quad := \bigvee_{((q,q') \overset{a_k}{\Rightarrow} [q_1,\ldots,q_k]) \in \Delta_2} \Big(\bigwedge_{i=1\ldots k} (\text{a\_dec } \mathcal{A} \; n_i \; q_i \; \varphi)\Big) \text{ , if } \varphi(a_k) = q'$$

$$\text{a\_dec } \mathcal{A} \;\; (\boldsymbol{\nu} a_k.n) \; q \; \varphi \quad := \bigvee_{(q \overset{a_k}{\Rightarrow} q') \in \Delta_3} (\text{a\_dec } \mathcal{A} \; n \; q' \; \varphi[a_k := q])$$

Correctness of a_dec

Theorem

$n \in \mathcal{L}(\mathcal{A}, q, \varphi) \leftrightarrow \text{a\_dec } \mathcal{A}\ n\ q\ \varphi = \text{true}$

- " $\Rightarrow$ " : Induction on $n \in \mathcal{L}(\mathcal{A}, q, \varphi)$
- " $\Leftarrow$ " : Induction on n
- Case analysis on $\varphi(-)$

Corollary

$\exists q, \varphi.\ n \in \mathcal{L}(\mathcal{A}, q, \varphi)$ is decidable

Next steps

- Show that denotational equality $[\![n]\!] = [\![n']\!]$ is decidable
  - Define a normal form for ν-trees [Gabbay and Ciancia, 2011] [Kozen et al., 2015]
- Show that emptiness is decidable for $\mathcal{L}(\mathcal{A}, q, \varphi)$
  - Compute a smallest accepted tree for all q and φ
- Write the thesis

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