An Almost Constructive Proof of Classical First-Order Completeness

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We give an almost constructive proof of the strong completeness of classical first-order predicate logic. The approach is mainly based on Herbelin and Ilik's [1] constructive and Schumm's [2] minimal analysis of Henkin's method [3]. As such, we give a fully constructive proof of model existence and conclude completeness using a consequence of Markov's principle. The results have been fully verified in the Coq proof assistant.

1 First-Order Predicate Logic

Following [4], we consider an implicative-universal first-order logic. The connectives of the object logic are marked with a dot, such as $\dot{\to}$, so they can be easily distinguished from their meta-logical counterparts. The parameters are required as we will be using a Gentzen-style natural deduction system.

$$
\begin{array}{rcll}
s, t : \mathsf{T} & ::= & e \mid f\ t \mid g\ t \mid x \mid p & x : \mathbf{N}, p : \mathbf{N} \quad \textbf{terms} \\
\varphi, \psi : \mathsf{F} & ::= & \dot{\bot} \mid P\ s\ t \mid \varphi \dot{\to} \psi \mid \dot{\forall} x.\varphi & x : \mathbf{N} \quad \textbf{formulas}
\end{array}
$$

We will write $\dot{\neg}\varphi$ for $\varphi \dot{\to} \dot{\bot}$ and $\dot{\exists} x.\varphi$ for $\dot{\neg}\dot{\forall} x.\dot{\neg}\varphi$. We define **theories** as predicates $\mathsf{F} \to \mathbf{P}$. If $\mathcal{T}\ \varphi$ we call φ a **element** of $\mathcal{T}$ and write $\varphi \in \mathcal{T}$. If all elements of $\mathcal{T}$ are elements of $\mathcal{T}'$, we say $\mathcal{T}$ **extends** $\mathcal{T}'$ and write $\mathcal{T} \subseteq \mathcal{T}'$. We write $\mathcal{T} \cup \{\varphi\}$ for $\lambda\psi.\psi \in \mathcal{T} \vee \varphi = \psi$.

Fact 1 The type F is **enumerable**. That is, there exists a surjective function $\mathcal{E}_{\mathsf{F}} : \mathbf{N} \to \mathsf{F}$. For every parameter $p : \mathbf{N}$ and $n \le p$, we know that p is fresh for $\mathcal{E}_{\mathsf{F}}\ n$.

Following Herbelin and Ilik [1], we use a Tarski semantic defined in terms of provability in our meta-logic. An **interpretation I on domain D** is characterized by $(e^{\mathbf{I}}, f^{\mathbf{I}}, g^{\mathbf{I}}, P^{\mathbf{I}}, \cdot^{\mathbf{I}})$. Together with an **assignment** $\rho : \mathbf{N} \to \mathbf{D}$, it gives rise to a **term interpretation** $\cdot^{\mathbf{I},\rho} : \mathsf{T} \to \mathbf{D}$ as defined below. We can now define a recursive semantic embedding into our meta-logic $\vDash_{\mathbf{I}}$: $(\mathbf{N} \to \mathbf{D}) \to \mathsf{F} \to \mathbf{P}$.

$$
\begin{aligned}
\_^{\mathbf{I},\rho} &: \mathsf{T} \to \mathbf{D} \\
e^{\mathbf{I},\rho} &= e^{\mathbf{I}} \\
(h\ t)^{\mathbf{I},\rho} &= h^{\mathbf{I}}\ t^{\mathbf{I},\rho} \\
x^{\mathbf{I},\rho} &= \rho\ x \\
p^{\mathbf{I},\rho} &= p^{\mathbf{I}}
\end{aligned}
\qquad
\begin{aligned}
\vDash_{\mathbf{I}}&: (\mathbf{N} \to \mathbf{D}) \to \mathsf{F} \to \mathbf{P} \\
\rho \vDash_{\mathbf{I}} \dot{\bot} &= \bot \\
\rho \vDash_{\mathbf{I}} P\ s\ t &= P^{\mathbf{I}}\ s^{\mathbf{I},\rho}\ t^{\mathbf{I},\rho} \\
\rho \vDash_{\mathbf{I}} \varphi \dot{\to} \psi &= \rho \vDash_{\mathbf{I}} \varphi \to \rho \vDash_{\mathbf{I}} \psi \\
\rho \vDash_{\mathbf{I}} \dot{\forall} x.\varphi &= \forall d : \mathbf{D}.\ \rho[x \leftarrow d] \vDash_{\mathbf{I}} \varphi
\end{aligned}
$$

We write $\rho \vDash_{\mathbf{I}} \mathcal{T}$ if $\rho \vDash_{\mathbf{I}} \varphi$ for all $\varphi \in \mathcal{T}$. We write $\mathcal{T} \vDash \varphi$ if for all $\mathbf{I}$ and ρ it holds that $\rho \vDash_{\mathbf{I}} \mathcal{T} \to \rho \vDash_{\mathbf{I}} \varphi$. Finally, if there exist $\mathbf{I}, \rho$ with $\rho \vDash_{\mathbf{I}} \mathcal{T}$ we say $\mathcal{T}$ **has a model**.

As we already mentioned above, we use a Gentzen-style classical natural deduction system. It is defined as the following inductive predicate $\vdash : \mathcal{L}(\mathsf{F}) \to \mathsf{F} \to \mathbf{P}$. Here φ^x_t denotes the formula obtained when substituting every free occurrence of x in φ with t.

$$
\textsc{Ctx}\ \frac{\varphi \in A}{A \vdash \varphi}
\qquad
\text{II}\ \frac{\varphi :: A \vdash \psi}{A \vdash \varphi \dot{\to} \psi}
\qquad
\text{IE}\ \frac{A \vdash \varphi \dot{\to} \psi \qquad A \vdash \varphi}{A \vdash \psi}
\qquad
\textsc{Exp}\ \frac{A \vdash \dot{\bot}}{A \vdash \varphi}
$$

$$
\text{DN}\ \frac{A \vdash \dot{\neg}\dot{\neg}\varphi}{A \vdash \varphi}
\qquad
\textsc{AllI}\ \frac{A \vdash \varphi^x_p \qquad p \text{ fresh for } \varphi \text{ and } A}{A \vdash \dot{\forall} x.\varphi}
\qquad
\textsc{AllE}\ \frac{A \vdash \dot{\forall} x.\varphi \qquad t \text{ closed}}{A \vdash \varphi^x_t}
$$

We also derive these additional rules.

$$
\textsc{ExI}\ \frac{A \vdash \varphi^x_t}{A \vdash \dot{\exists} x.\varphi}
\qquad
\textsc{ExE}\ \frac{A \vdash \dot{\exists} x.\varphi \qquad \varphi^x_p :: A \vdash \psi \qquad p \text{ fresh for } \varphi, \psi \text{ and } A}{A \vdash \psi}
$$

If every element of A is an element of $\mathcal{T}$, we write as $A \subseteq \mathcal{T}$. If there exists an $A \subseteq \mathcal{T}$ such that $A \vdash \varphi$, we say $\mathcal{T}$ **entails** φ and write $\mathcal{T} \vdash \varphi$.

Fact 2 $\mathcal{T} \cup \{\varphi\} \vdash \psi \to \mathcal{T} \vdash \varphi \to \mathcal{T} \vdash \psi$

Lemma 3 (Refutation completeness) $\mathcal{T} \vdash \varphi \leftrightarrow \mathcal{T} \cup \{\dot{\neg}\varphi\} \vdash \dot{\bot}$

The notion of consistency is crucial for this proof, as the model will be constructed by extending a consistent theory into a maximally consistent one. We call a theory $\mathcal{T}$ **consistent** if it is impossible to derive a contradiction from it, that is, if $\mathcal{T} \nvdash \dot{\bot}$. A consistent $\mathcal{T}$ is **maximally consistent** if for all φ with $\mathcal{T} \cup \{\varphi\}$ consistent, φ already is an element of $\mathcal{T}$.

Lemma 4 Let $\mathcal{T}$ be a theory and φ be a formula such that $\mathcal{T} \vdash \varphi$.

1. If $\mathcal{T}$ is consistent then so is $\mathcal{T} \cup \{\varphi\}$.
2. If $\mathcal{T}$ is maximally consistent, then φ is an element of $\mathcal{T}$.

2 The Lindenbaum lemma

The Lindenbaum construction completes a consistent theory into a maximally consistent theory. The intuition behind this construction is that we simply scan through all formulas and add all those that maintain consistency.

Definition 5 (Consistent extension) The consistent extension of $\mathcal{T}$ and φ is

$$\mathcal{T} \cup\!\!\!? \{\varphi\} := \lambda\psi.\psi \in \mathcal{T} \vee (\varphi = \psi \wedge \mathcal{T} \cup \{\varphi\} \text{ is consistent})$$

Lemma 6 For any $\mathcal{T}, \varphi, \psi$ with $\mathcal{T} \cup\!\!\!? \{\varphi\} \vdash \psi$ it holds that $\mathcal{T} \vdash \psi$ or $\mathcal{T} \cup\!\!\!? \{\varphi\}$ is consistent.

Definition 7 (Omega) Let $\mathcal{T}$ be a theory. Then define Ω as follows:

$$\Omega_0 := \mathcal{T} \qquad \Omega_{n+1} := \Omega_n \cup\!\!\!? \{\mathcal{E}_{\mathsf{F}}\ n\} \qquad \Omega := \lambda\varphi.\ \exists n.\ \varphi \in \Omega_n$$

Lemma 8 Let $\mathcal{T}$ be consistent.

1. Ω_n is consistent for every n.
2. Ω is maximally consistent.

3 Henkin axioms

The next construction is required specifically to allow us to prove that our model satisfies universally quantified formulas. Intuitively, we create closed witnesses for the validity of all universally quantified formulas.

Definition 9 (Henkin theories) Let $\mathcal{T}$ be a theory. We call $\mathcal{T}$ Henkin if for all x and φ, there exists a closed term t such that $\varphi_t^x \in \mathcal{T} \rightarrow \dot\forall x.\varphi \in \mathcal{T}$.

Definition 10 (Henkin axioms) Let $\mathcal{T}$ be a theory. Then define $\mathcal{H}$ as follows:

$$\mathcal{H} := \lambda\varphi.\ \exists n.\ \varphi \in \mathcal{H}_n \qquad \mathcal{H}_0 = \mathcal{T} \qquad \mathcal{H}_{n+1} = \begin{cases} \mathcal{H}_n \cup \{\varphi_n^x \dot\rightarrow \dot\forall x.\varphi\} & \text{if } \mathcal{E}_{\mathsf{F}}\ n = \dot\forall x.\varphi \\ \mathcal{H}_n & \text{otherwise} \end{cases}$$

Lemma 11 Let $\mathcal{T}$ be a parameter-free consistent theory.

1. p is fresh for any $\varphi \in \mathcal{H}_p$.
2. $\mathcal{H}_n$ is consistent for any n.
3. $\mathcal{H}$ is consistent and Henkin.

Proof 2. Per induction, we have to contradict $\mathcal{H}_n \vdash \dot\neg(\varphi_n^x \dot\rightarrow \dot\forall x.\varphi)$. As n is fresh for $\mathcal{H}_n$ and $\varphi_n^x \dot\rightarrow \dot\forall x.\varphi$ can be obtained from the drinker's paradox $\dot\exists y.\varphi_y^x \dot\rightarrow \dot\forall x.\varphi$ we can use ExE to conclude $\mathcal{H}_n \vdash \bot$ and derive the contradiction. ∎

4 Herbrand Model

In this section we construct a model for any closed subset of a theory that is maximally consistent and Henkin. Per sections 2 and 3, this suffices for constructing a model of any consistent, closed and parameter-free theory. More specifically, we construct a **Herbrand model**, that is, a model whose domain are the closed terms.

Definition 12 (Herbrand model) Let $\mathcal{T}$ be a theory. The Herbrand model on $\mathcal{T}$ is given by the following interpretation in the domain of closed terms T^c:

$$e^{\mathcal{T}} := e \qquad p^{\mathcal{T}} := p \qquad f^{\mathcal{T}}\ x := f\ x \qquad g^{\mathcal{T}}\ x := g\ x \qquad P^{\mathcal{T}}\ s\ t = P\ s\ t \in \mathcal{T}$$

Lemma 13 Let $\rho : \mathbf{N} \to \mathsf{T}^c$ be an assignment. $\forall x, (t : \mathsf{T}^c).\rho[x \to t] \vDash_{\mathcal{T}} \varphi \leftrightarrow \rho \vDash_{\mathcal{T}} \varphi^x_t$.

Lemma 14 (Theory closure) Let $\mathcal{T}$ be a maximally consistent theory.

1. For formulas φ and ψ it holds that $\varphi \dot{\to} \psi \in \mathcal{T}$ if and only if $\varphi \in \mathcal{T} \to \psi \in \mathcal{T}$.
2. If $\mathcal{T}$ is Henkin, then for any φ and x it holds that $\dot{\forall} x.\varphi \in \mathcal{T}$ if and only if $\forall t : \mathsf{T}^c.\varphi^x_t \in \mathcal{T}$.

Proof 1. $\to$ Follows per Lemma 4 and IE.

$\leftarrow$ Using Lemma 4, we contradict $\mathcal{T} \cup \{\varphi \dot{\to} \psi\} \vdash \dot{\bot}$ by showing that $\mathcal{T} \vdash \varphi$ and $\mathcal{T} \vdash \dot{\neg}\psi$. Per assumption $\mathcal{T} \vdash \psi$ holds as well which is contradictory.

2. $\to$ Per Lemma 4 and AllE, we can show that $\varphi^x_t \in \mathcal{T}$ for any closed t.

$\leftarrow$ As $\mathcal{T}$ is Henkin, there is a closed term h, such that $\varphi^x_h \dot{\to} \dot{\forall} x.\varphi \in \mathcal{T}$. As h is closed, $\varphi^x_h \in \mathcal{T}$ per IH. Then per Lemma 4 and IE, $\dot{\forall} x.\varphi \in \mathcal{T}$. ∎

Lemma 15 (Model correctness) Let $\mathcal{T}$ be a theory both maximally consistent and Henkin.

1. For any assignment ρ and any closed formula φ it holds that $\varphi \in \mathcal{T}$ if and only if $\rho \vDash_{\mathbf{I}} \varphi$.
2. For any closed theory $\mathcal{T}' \subseteq \mathcal{T}$ it holds that $\vDash_{\mathcal{T}} \mathcal{T}'$.
3. $\mathcal{T}$ is a classical model for closed terms. That is, for every closed φ, $\Vdash_{\mathcal{T}} \neg\neg\varphi \dot{\to} \varphi$.

Proof 1. Follows from Lemma 14 per size induction on φ.

2, 3. Follow from 1. ∎

By looking at the proofs of Lemma 14 and Lemma 15 one can discern why the restrictions on $\mathcal{T}$ and φ were necessary: We need to restrict our model to the closed terms as only then Lemma 13 applies and the AllE rule can also only be instantiated with closed terms, making the choice of closed terms for the domain crucial to proving Lemma 15 for the universal quantifier. The necessity for φ to be closed becomes apparent in the base case as $P\ s^{\mathbf{I},\rho}\ t^{\mathbf{I},\rho} = P\ s\ t$ is needed, which only holds under this condition.

5 Completeness

Using these constructions, we can prove the model existence theorem. Note that all proofs up to now have been fully constructive. Deducing strong completeness requires Assumption 18, which is a consequence of Markov's principle, regarded by some schools of thought as constructive. We do not share this view as it is not provable in Coq (hence the "almost constructive proof").

Theorem 16 (Model existence) Any consistent, closed, and parameter-free theory has a model.

Proof Let $\mathcal{T}$ be such a theory. Per Lemma 11, there is a consistent extension $\mathcal{T} \subseteq \mathcal{H}$ which is Henkin. This theory can be per Lemma 8 extended to a maximally complete theory $\mathcal{H} \subseteq \Omega$ which is still Henkin. As $\mathcal{T} \subseteq \Omega$ is closed, we can now conclude $\vDash_\Omega \mathcal{T}$ per Lemma 15. ■

Theorem 17 (Strong quasi-completeness) Let $\mathcal{T}, \varphi$ be closed and parameter-free. Then $\mathcal{T} \vDash \varphi \to \neg\neg\mathcal{T} \vdash \varphi$.

Proof Assume $\mathcal{T} \vDash \varphi$. Per Lemma 3 is suffices to contradict the consistency of $\mathcal{T} \cup \{\neg\varphi\}$. But per Theorem 16, we get a model $\mathbf{I}$ such that $\vDash_\mathbf{I} \mathcal{T} \cup \{\neg\varphi\}$ and especially $\vDash_\mathbf{I} \neg\varphi$. From $\mathcal{T} \vDash \varphi$ we can also conclude that $\vDash_\mathbf{I} \varphi$. This is a contradiction. ■

Assumption 18 (Stability of $\vdash$) For any enumerable $\mathcal{T}$ and any φ, $\neg\neg\mathcal{T} \vdash \varphi \to \mathcal{T} \vdash \varphi$.

Theorem 19 (Strong completeness) Let $\mathcal{T}, \varphi$ be closed and parameter-free.

1. If $\mathcal{T}$ is stable, that is $\forall\varphi, \neg\neg\mathcal{T} \vdash \varphi \to \mathcal{T} \vdash \varphi$, then $\mathcal{T} \vDash \varphi \to \mathcal{T} \vdash \varphi$.
2. If Assumption 18 holds then $\mathcal{T} \vDash \varphi \to \mathcal{T} \vdash \varphi$ for any enumerable $\mathcal{T}$.

References

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