Soundness and Completeness of Intuitionistic Dialogues

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We give a proof of soundness and completeness of generalized intuitionistic E-dialogues by adapting the approach of Sørensen and Urzyczyn [1] to a novel style of formalization for dialogues. We then extend the result to D-dialogues and demonstrate how to obtain soundness and completeness between dialogues and the normal sequent calculus LJT for the universal-implicative fragment of first-order logic. Unless stated otherwise, these results have been formalized in the Coq interactive proof assistant.

1 Dialogical Logic

Dialogical logic, or dialogue games, characterizes a formula as valid if it can always be defended in a debate with another party seeking to dismantle the claim. This sets it apart from other semantic approaches, such as models and derivations, because it explicitly founds on the debate centered roots of logical truth. This section will briefly introduce dialogues for intuitionistic first-order logic through the example of D-dialogues in the style of Felscher [2].

Dialogues take the form of a two-player turn-taking game, represented by a sequence of moves. During the game, each player attacks formulae admitted by their opponent and defends against attacks on their own admissions. Figure 1 describes which attacks may be leveled against which formula and how they can be defended against, which is known as the **logical rules**.

Formula	Attack	Defenses	Formula	Attack	Defenses
$\bot$	$A_\bot$	—	$\top$	—	—
$\varphi \rightarrow \psi$	$A_\rightarrow\ \varphi$	$\{\psi\}$	$\varphi \vee \psi$	$A_\vee$	$\{\varphi, \psi\}$
$\varphi \wedge \psi$	A_L	$\{\varphi\}$	$\forall\varphi$	A_t	$\{\varphi[t]\}$
$\varphi \wedge \psi$	A_R	$\{\psi\}$	$\exists\varphi$	$A_\exists$	$\{\varphi[t] \mid t : \mathsf{T}\}$

Figure 1: Local rules for intuitionistic first-order logic

A D-dialogue starts by the proponent admitting the formula that is to be validated. The two players then take turns, making one move each. They can choose between attacking a formula the other player has admitted previously in the dialogue or defending against the challenge, the last attack against their admissions they have not yet defended against, by admitting one of the defense formulae. Note that a player attacking an implication is also forced to admit its premise. There are additional restrictions placed upon the moves that can be performed: The opponent may attack each of the proponent's admissions only once. In turn, the proponent may only admit an atomic formula if it has already been admitted by the opponent. A dialogue is won by the proponent if it is finite and ends in a proponent move. These rules are called the **structural rules**.

The asymmetry of the structural rules align with the goals of the players. As the proponent is trying to defend the formula, there only needs to be a restriction on what formulas she can admit. On the other hand, as the opponent aims to prevent the proponent from winning, she has to be prevented from stalling the game indefinitely by continuously attacking the proponent's admissions. A formula is considered valid if the proponent can always win the dialogue. This is formalized by the existence of a **winning strategy**, which can be represented as a tree as pictured in Figure 2.

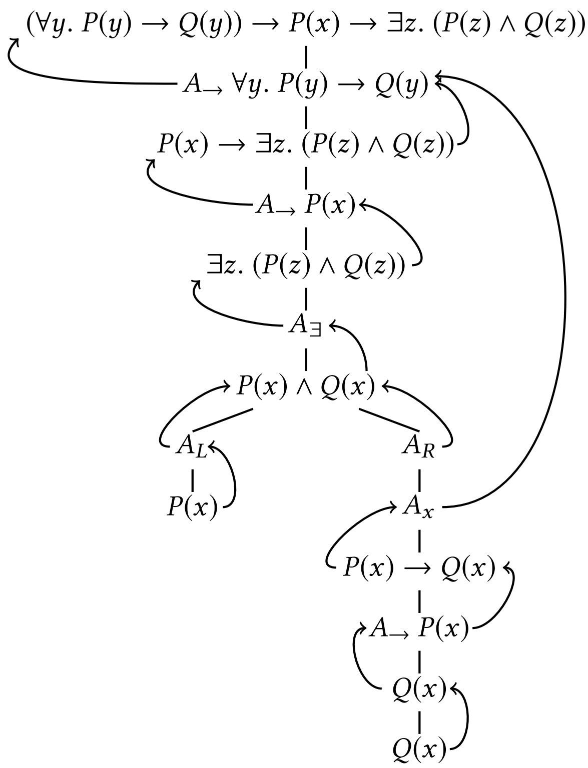

Figure 2: A winning D-strategy

2 Generalized Intuitionistic E-Dialogues

In [1], Sørensen and Urzyczyn prove the soundness and completeness of classical E-dialogues, generalized over the local rules, with regards to two sequent calculi called LKD and LKd. In this section, we will adopt their techniques to show soundness and completeness of generalized intuitionistic E-dialogues with regards to LJD, the intuitionistic variant of LKD. We will also introduce a new way of formalizing dialogues as state transition systems. This will allow us state the correspondence between derivations in LJD and winning dialogue strategies more directly and is better suited for formalizations in type theory overall.

A **rule set** $(\mathsf{F}, \mathsf{F}^a, \mathcal{A}, (\mathcal{D}_a)_{a\in\mathcal{A}}, \lhd)$ of local rules consists of a collection of formulae F, a subcollection of formulae F^a considered **atomic**, a collection of **attacks** $\mathcal{A}$, the **defenses** $(\mathcal{D}_a)_{a\in\mathcal{A}}$, a family of collections of formulae indexed by the attacks, and the **attacks relation** $\lhd : \mathsf{F} \to \mathcal{A} \to \mathcal{O}(\mathsf{F}) \to \mathbf{P}$. Whenever $\varphi \lhd a \mid \psi$, we say "a is an **attack on** φ" and call ψ the **admission**. If $\varphi \in \mathcal{D}_a$, we call φ a **defense against** a.

We define the intuitionistic E-dialogues in terms of rule sets. They differ from the D-dialogues introduced in Section 1 by a restriction imposed on the opponent: she can only react to the preceding proponent move. However, their proving strength is equivalent to D-dialogues of the same rule set, as we will show in Section 3. As opposed to the word- and tree-based formalization usually employed throughout the literature, we will formalize them as a state transition system. The states are given by the collection $\mathcal{L}(\mathsf{F}) \times \mathcal{A}$ of pairs of the opponent's previous admissions A_o and current challenge against the proponent c. The definitions of the transition rules are given in Figure 3. The relation $s \rightsquigarrow_p m$ states that the proponent can take move m in state s. A move is either an attack on the opponent's admissions $PA\ a$ or a defense against the challenge $PD\ \varphi$. The predicate "justified $A_o\ \varphi$" is defined as $\varphi \in \mathsf{F}^a \to \varphi \in A_o$ and is used to assert that the proponent only utters atomic formulae the opponent has already admitted. The relation $s\,; m \rightsquigarrow_o s'$ attests that the opponent can make a move reacting to a proponent's move m in state s that leads to state s'. The usual winning property of dialogue games (the existence of a strategy tree in which every maximal path is of finite length and ends with a proponent move) can now be restated as a variant of the strong normalization of the transition system, in which only one proponent move is considered at every step. We now say that a formula φ is **E-valid** if φ is not atomic and for every attack $\varphi \lhd c \mid \psi$, Win $([\psi], c)$ holds.

The remarkable idea behind the approach of Sørensen and Urzyczyn [1] is the formulation of LKD, a sequent calculus that is isomorphic to the winning strategies of classical E-dialogues. We will adapt this into an intuitionistic variant LJD of the same nature laid out in Figure 4. It has multiple notable properties: Firstly, as it has the signature $\vdash : \mathcal{L}(\mathsf{F}) \to 2^{\mathsf{F}} \to \mathbf{P}$, its consequences are possibly infinite subcollections of the formulae. This is fundamentally different from the usual intuitionistic sequent calculus LJ which is restricted to at most one conclusion. Additionally, it is also possibly infinitely branching

$$\frac{\varphi \in A_o \quad \varphi \lhd a \mid \psi \quad \text{justified } A_o\, \psi}{(A_o, c) \rightsquigarrow_p PA\, a} \qquad \frac{\varphi \in \mathcal{D}_c \quad \text{justified } A_o\, \varphi}{(A_o, c) \rightsquigarrow_p PD\, \varphi}$$

$$\frac{\varphi \lhd c' \mid \psi}{(A_o, c)\, ;PD\, \varphi \rightsquigarrow_o (\psi :: A_o, c')} \qquad \frac{\varphi \in \mathcal{D}_a}{(A_o, c)\, ;PA\, a \rightsquigarrow_o (\varphi :: A_o, c)}$$

$$\frac{\varphi \lhd a \mid \ulcorner \psi \urcorner \quad \psi \lhd c' \mid \tau}{(A_o, c)\, ;PA\, a \rightsquigarrow_o (\tau :: A_o, c')}$$

$$\frac{s \rightsquigarrow_p m \quad \forall s'.\, s\, ;m \rightsquigarrow_o s' \longrightarrow \text{Win } s'}{\text{Win } s}$$

Figure 3: Rules for E-dialogues

as an attack might have infinitely many defenses or there might be infinitely many attacks on a formula, as is the case for $\exists\varphi$ and $\forall\varphi$ respectively. As stated earlier, derivations in LJD are isomorphic to winning strategies for E-dialogues. Intuitively, each of its two rules corresponds to one move the proponent can take: The L-rule corresponds to the proponent attacking an admission of the opponent and the R-rule to the proponent defending against current challenge.

$$\text{L}\,\frac{\varphi \in \Gamma \quad \text{justified } \Gamma\, \psi \quad \varphi \lhd a \mid \psi \quad \forall \sigma \in \mathcal{D}_a.\, \Gamma, \sigma \vdash \Delta \quad \forall \psi \lhd a' \mid \tau.\, \Gamma, \tau \vdash \mathcal{D}_{a'}}{\Gamma \vdash \Delta}$$

$$\text{R}\,\frac{\varphi \in \Delta \quad \text{justified } \Gamma\, \varphi \quad \forall \varphi \lhd a \mid \psi.\, \Gamma, \psi \vdash \mathcal{D}_a}{\Gamma \vdash \Delta}$$

Figure 4: System LJD

We are now ready to prove the results of this section, the soundness and completeness of LJD with regards to E-dialogues.

Theorem 1 (E-Soundness) If $\vdash \varphi$ then φ is E-valid.

Proof Prove $\forall \Gamma, \Delta.\ \Gamma \vdash \Delta \longrightarrow \forall c.\ \Delta \subseteq \mathcal{D}_c \longrightarrow \text{Win}\,(\Gamma, c)$ per induction on the derivation. Then the claim follows as $\vdash \varphi$ can only be proven through the R-rule. ∎

Theorem 2 (E-Completeness) If φ is E-valid then $\vdash \varphi$.

Proof Prove $\forall A_o, c.\ \text{Win}\ (A_o, c) \rightarrow A_o \vdash \mathcal{D}_c$ by induction on the strategy. Then the claim follows by application of the R-rule. ∎

3 Generalized Intuitionistic D-Dialogues

In this section, we will prove the soundness and completeness between LJD and generalized intuitionistic D-dialogues. For the soundness proof, we introduce a new kind of dialogue, the S-dialogues.

Because D-dialogues allow the opponent to react to proponent moves beyond the last one, their formalization is more complex than that of E-games. The states of the dialogues are given by the collection $\mathcal{L}(\mathsf{F}) \times \mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathsf{F}) \times \mathcal{L}(\mathcal{A})$. The first two components represent the formulae admitted by the proponent and the attacks she has not yet defended against, respectively. The last two components describe the same for the opponent. The rules are given in Figure 5. Analogously to E-dialogues, a formula φ is considered **D-valid** if it is not atomic and for every attack $\varphi \lhd c \mid \psi$, $\text{Win}\ (\mathit{nil}, [c], [\psi], \mathit{nil})$ holds.

$$\frac{\varphi \in \mathcal{D}_c \quad \text{justified}\ A_o\ \varphi}{(A_p, c :: C_p, A_o, C_o) \rightsquigarrow_p (\varphi :: A_p, C_p, A_o, C_o)}$$

$$\frac{\varphi \in A_o \quad \text{justified}\ A_o\ \psi \quad \varphi \lhd a \mid \psi}{(A_p, C_p, A_o, C_o) \rightsquigarrow_p (\psi :: A_p, C_p, A_o, a :: C_o)}$$

$$\frac{\varphi \in \mathcal{D}_a}{(A_p, C_p, A_o, a :: C_o) \rightsquigarrow_o (A_p, C_p, \varphi :: A_o, C_o)}$$

$$\frac{\varphi \lhd c \mid \psi}{(A_p +\!\!+ \varphi :: A'_p, C_p, A_o, C_o) \rightsquigarrow_o (A_p +\!\!+ A'_p, c :: C_p, \psi :: A_o, C_o)}$$

$$\frac{s \rightsquigarrow_p s' \quad \forall s''.\ s' \rightsquigarrow_o s'' \rightarrow \text{Win}\ s''}{\text{Win}\ s}$$

Figure 5: Rules for D-dialogues

Theorem 3 (D-Completeness) If φ is D-valid then $\vdash \varphi$.

Proof Prove $\forall A_p, C_p, A_o, C_o, c.\ \text{Win}\ (A_p, c :: C_p, A_o, C_o) \rightarrow A_o \vdash \mathcal{D}_c$ by induction on the strategy. Then the claim follows by application of the R-rule. ∎

The proof of the completeness of LJD with regards to D-dialogues requires some additional machinery. This is because its state space does not lend itself well to an invariant expressing the relationship between C_p and C_o in terms of LJD. For this purpose we introduce a new kind of dialogue, the S-dialogue ("S" as in "stack"). Its states are given by the collection $\mathcal{L}(\mathsf{F}) \times \mathcal{L}(\mathsf{F}) \times \mathcal{L}(\mathcal{A} \times \mathcal{A})$. The first two components are the proponent's and opponent's admissions. The third component combines the pending attacks C_p and C_o into one stack D of deferred moves. The rules are given in Figure 6. A formula φ is **S-valid** if it is not atomic and for every attack $\varphi \lhd c \mid \psi$, Win $(nil, [\psi], nil)\ c$ holds.

S-dialogues are motivated by an insight about how D-strategies that were extracted from LJD derivations can handle the current challenge: If the derivation ends in the R-rule, the proponent will defend against the challenge, thereby taking care of it. If it ends in the L-rule, the proponent will attack one of the opponent's admissions. However, the information contained within the subderivation only describes how to proceed fending off the challenge once the opponent has chosen to defend against the attack, which defers the further moves prescribed by the derivation indefinitely. This is represented in the formalization by putting the proponent's attack and the challenge being on the stack of deferred moves. The opponent will then always introduce a new challenge, either by defending against the last deferred proponent attack, thereby returning a deferred challenge into focus or attacking one of the proponent's admissions.

$$\frac{\varphi \in \mathcal{D}_c \quad \text{justified } A_o\ \varphi}{(A_p, A_o, D)\ ;c \rightsquigarrow_p (\varphi :: A_p, A_o, D)} \qquad \frac{\varphi \in A_o \quad \text{justified } A_o\ \psi \quad \varphi \lhd a \mid \psi}{(A_p, A_o, D)\ ;c \rightsquigarrow_p (\psi :: A_p, A_o, (a, c) :: D)}$$

$$\frac{\varphi \in \mathcal{D}_a}{(A_p, A_o, (a, c) :: D) \rightsquigarrow_o (A_p, \varphi :: A_o, D)\ ;c}$$

$$\frac{\varphi \lhd c \mid \psi}{(A_p +\!\!+ \varphi :: A_p', A_o, D) \rightsquigarrow_o (A_p +\!\!+ A_p', \psi :: A_o, D)\ ;c}$$

$$\frac{s\ ;c \rightsquigarrow_p s' \quad \forall s'', c'.\ s' \rightsquigarrow_o s''\ ;c' \longrightarrow \text{Win } s''\ c}{\text{Win } s\ c}$$

Figure 6: Rules for S-dialogues

Theorem 4 (S-Soundness) If $\vdash \varphi$ then φ is S-valid.

Proof Prove
$$\begin{aligned}\forall A_P, A_o, D, c.\ &(\forall \varphi \in A_p, \varphi \lhd a \mid \psi . \psi :: A_o \vdash \mathcal{D}_a) \longrightarrow\\ &(\forall (a, c) \in D, \varphi \in D_a.\ \varphi :: A_o \vdash \mathcal{D}_c) \longrightarrow\\ &A_o \vdash \mathcal{D}_c \longrightarrow \text{Win } (A_p, A_o, D)\ c\end{aligned}$$

by induction on all of the derivations. The result then follows as $\varphi \vdash$ can only be proven through the R-rule.

Note that because of its complicated induction scheme, this proof has not been formalized in Coq yet. ∎

Theorem 5 (D-Embedding) If φ is S-valid then it is D-valid.

Proof Prove $\forall A_p, A_o, D, c.\ \text{Win}\ (A_p, A_o, D)\ c \rightarrow \text{Win}\ (A_p, c :: C_p, A_o, C_o)$ per induction on the strategy with $D = [(o_1, p_1), \ldots, (o_n, p_n)]$ and $C_x = [x_1, \ldots, x_n]$. The result then follows from the respective definitions of validity. ∎

Corollary 6 (D-Soundness) If $\vdash \varphi$ then φ is D-valid.

Corollary 7 (D-E-Equivalence) φ is D-valid if and only if it is E-valid.

4 Soundness and Completeness between Dialogues and LJT

In this section we will demonstrate the usefulness of these results and connect them to results obtained prior by using them to derive soundness and completeness for the intuitionistic normal universal-implicative sequent calculus LJT with regards to intuitionistic dialogues.

The proofs will be arrived at by translating LJT into LJD and LJD into intuitionistic ND, which were defined and shown equivalent in [3]. LJD is not well suited for a direct translation back into LJT because proofs in LJD are not necessarily normal as the L-rule can be used freely. We first show a few general properties about LJD which help the translation. The logical rules used in the translation are given in Figure 7, predicates being the only formulae considered atomic. It is worth noting that this result can be extended to the full intuitionistic sequent calculus LJ in a straightforward manner.

Formula	Attack	Defenses
$\bot$	$A_\bot$	—
$\varphi \rightarrow \psi$	$A_\rightarrow\ \varphi$	$\{\psi\}$
$\forall\varphi$	A_t	$\{\varphi[t]\}$

Figure 7: Logical rules of the translation

Lemma 8 (Properties of LJD)

1. Let $\Gamma \vdash \Delta$ with $\Gamma \subseteq \Gamma', \Delta \subseteq \Delta'$. Then $\Gamma' \vdash \Delta'$.
2. Let $\Gamma \vdash \{\varphi\}$. Then for every $\varphi \lhd a \mid \psi$ we have that $\Gamma \vdash \mathcal{D}_a$.

3. Let $\preccurlyeq$ be a well founded relation such that $\psi \preccurlyeq \varphi$ and $\forall \sigma \in \mathcal{D}_a.\ \sigma \preccurlyeq \varphi$ for any $\varphi \lhd a \mid \psi$. Then $\Gamma \vdash \Delta$ whenever $\varphi \in \Gamma$ and $\varphi \in \Delta$ for any φ.
4. Let $\Gamma \vdash \{\varphi\}$ and $(\forall \Gamma \subseteq \Gamma'.\text{justified}\ \Gamma'\ \varphi \rightarrow \Gamma' \vdash \Delta)$. Then $\Gamma \vdash \Delta$.

Proof 1,2,4. Per induction on the derivation.

3. Per well founded induction on the relation $\preccurlyeq$. ■

Theorem 9 (Embedding LJT into LJD) Let $\vdash_{LJ} \Gamma\ \psi\ \varphi$. Then $\Gamma, \psi \vdash \varphi$.

Proof Prove $\forall \Gamma, \psi, \varphi, \rho.\ \ \vdash_{LJ} \Gamma\ \psi\ \varphi \rightarrow (\Gamma, \psi)[\rho] \vdash \varphi[\rho]$ per induction on the derivation using the properties from Lemma 8. ■

Theorem 10 (Embedding LJD into ND) Let $\Gamma \vdash \{\varphi\}$. Then $\Gamma \vdash_{ND} \varphi$.

Proof Prove $\forall \Gamma, \Delta.\ \Gamma \vdash \Delta \rightarrow (\exists \varphi \in \Delta.\ \Gamma \vdash_{ND} \varphi) \lor (\forall \varphi.\ \Gamma \vdash_{ND} \varphi)$ per induction on the derivation. ■

Corollary 11 LJT on the universal-implicative fragment is sound and complete with regards to intuitionistic E- and D-dialogues.

References

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