A Constructive Analysis of First-Order Completeness Theorems in Coq

Bachelor Talk

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`https://www.ps.uni-saarland.de/~wehr/bachelor.php`

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Overview

Stability of Deduction

A proposition P is called **stable** if $\neg\neg P \to P$.

The principle of C-**stability** is $\forall \mathcal{T}, \varphi.\ C\,\mathcal{T} \to \mathsf{stable}(\mathcal{T} \vdash \varphi)$.

Class	**Principle**
$\lambda\mathcal{T}.\ \top$	$\forall P.\ \mathsf{stable}\,P$
$\lambda\mathcal{T}.\ \mathcal{T}$ is enumerable	$\forall f : \mathbb{N} \to \mathbb{B}.\ \mathsf{stable}\,(\exists n.\ f\,n = \mathsf{true})$
$\lambda\mathcal{T}.\ \mathcal{T}$ is finite	$\forall s.\ \mathsf{stable}\,(\exists t.\ s \rhd t)$

Constructive Analysis

$$\frac{\text{Completeness} \longleftrightarrow C\text{-Stability}}{\text{CIC}}$$

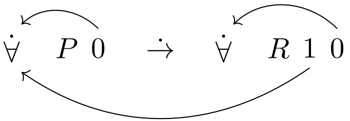
$$\text{CIC} \nvdash C\text{-Stability}$$

Overview

First-Order Fragment

$$t : \mathfrak{T} \;::=\; x \mid f\,t_1 \ldots t_{|f|} \qquad\qquad f : \mathcal{F},\; x : \mathbb{N} \qquad \textbf{terms}$$

$$\varphi, \psi : \mathfrak{F}_F \;::=\; \dot{\bot} \mid P\,t_1 \ldots t_{|P|} \mid \varphi \dot{\to} \psi \mid \dot{\forall}\varphi \qquad\qquad P : \mathcal{P} \qquad \textbf{formulas}$$



Models

An **interpretation on a domain** D consists of

$$f^{\mathcal{I}} : D^{|f|} \to D \qquad P^{\mathcal{I}} : D^{|P|} \to \mathbb{P} \qquad \bot^{\mathcal{I}} : \mathbb{P}$$

Given $\rho : \mathbb{N} \to D$, we define $t^\rho : D$ and $\rho \vDash \varphi \ : \ \mathbb{P}$

$$\begin{aligned} \rho \vDash \dot{\bot} &:= \bot^{\mathcal{I}} \\ \rho \vDash P\, t_1 \dots t_{|P|} &:= P^{\mathcal{I}}\, t_1^\rho \dots t_{|P|}^\rho \\ \rho \vDash \varphi \mathrel{\dot{\to}} \psi &:= \rho \vDash \varphi \ \to \ \rho \vDash \psi \\ \rho \vDash \dot{\forall} \varphi &:= \forall d : D.\ d, \rho \vDash \varphi \end{aligned}$$

Non-standard Models

Consider an interpretation on some domain D with

$$P^{\mathcal{I}}\, v := \top \qquad \bot^{\mathcal{I}} := \top$$

Then for any $\rho : \mathbb{N} \to D$

$$\rho \vDash \varphi \qquad \text{and} \qquad \rho \vDash \dot{\neg}\varphi$$

Non-standard Models

Consider an interpretation on some domain D with

$$P^{\mathcal{I}}\, v := \top \qquad \bot^{\mathcal{I}} := \top$$

Then for any $\rho : \mathbb{N} \to D$

$$\rho \vDash \varphi \qquad \text{and} \qquad \rho \vDash \dot{\neg}\varphi$$

An interpretation I has a **standard** $\dot{\bot}$ if

$$\bot^{\mathcal{I}} \to \bot$$

Constrained Validity

Given a constraint $X : \mathcal{I} \to \mathbb{P}$ we define the **validity of a formula**

$$\mathcal{T} \vDash_X \varphi := \forall I : \mathcal{I}, \rho.\ X\, I \to \rho \vDash \mathcal{T} \to \rho \vDash \varphi$$

We define the constraint of **standard models** as

$$S\, I := \bot^{\mathcal{I}} \to \bot \ \wedge\ \forall \rho, \varphi, \psi.\ \rho \vDash (((\varphi \dot{\to} \psi) \dot{\to} \varphi) \dot{\to} \varphi)$$

Non-Standard Models

We define the constraint of **exploding models** as

$$E\,I := (\forall \rho, \varphi.\ \bot^{\mathcal{I}} \to \rho \vDash \varphi)\ \wedge\ \forall \rho, \varphi, \psi.\ \rho \vDash (((\varphi \dot{\to} \psi) \dot{\to} \varphi) \dot{\to} \varphi)$$

Non-Standard Models

We define the constraint of **exploding models** as

$$E\,I := (\forall \rho, \varphi.\ \bot^{\mathcal{I}} \to \rho \vDash \varphi)\ \wedge\ \forall \rho, \varphi, \psi.\ \rho \vDash (((\varphi \dot{\to} \psi) \dot{\to} \varphi) \dot{\to} \varphi)$$

We define the constraint of **minimal models** as

$$M\,I := \forall \rho, \varphi, \psi.\ \rho \vDash_X (((\varphi \dot{\to} \psi) \dot{\to} \varphi) \dot{\to} \varphi)$$

Natural Deduction

$$\Gamma \vdash_{SB} \varphi$$

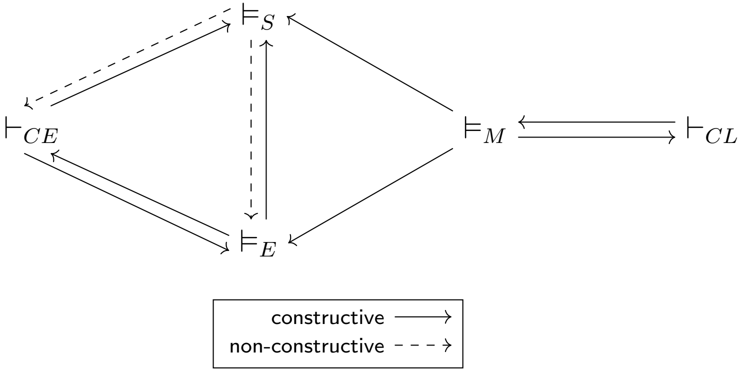
$$\textsc{Ctx}\,\frac{\varphi \in \Gamma}{\Gamma \vdash \varphi} \qquad \text{II}\,\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \dot{\to} \psi} \qquad \text{IE}\,\frac{\Gamma \vdash \varphi \dot{\to} \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$$

$$\textsc{AllI}\,\frac{\uparrow\Gamma \vdash \varphi}{\Gamma \vdash \dot{\forall}\varphi} \qquad \textsc{AllE}\,\frac{\Gamma \vdash \dot{\forall}\varphi}{\Gamma \vdash \varphi[t]} \qquad \textsc{Exp}\,\frac{\Gamma \vdash_E \dot{\bot}}{\Gamma \vdash_E \varphi}$$

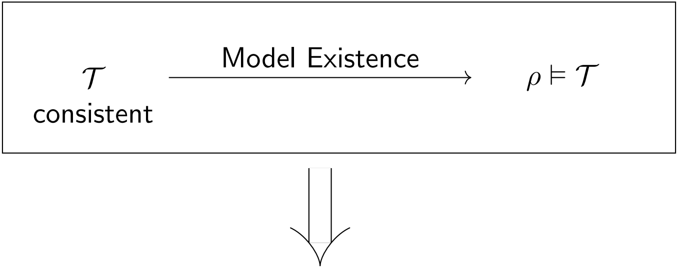
$$\textsc{Peirce}\,\frac{}{\Gamma \vdash_C ((\varphi \dot{\to} \psi) \dot{\to} \varphi) \dot{\to} \varphi}$$

$$\mathcal{T} \vdash_{SB} \varphi := \exists \Gamma \subseteq \mathcal{T}.\ \Gamma \vdash_{SB} \varphi$$

Tarski Results



Proof Outline



$$\mathcal{T} \vDash_S \varphi \to \mathcal{T} \vdash_{CE} \varphi$$

Proof Outline

$\mathcal{T}$ consistent $\xrightarrow{\text{Model Existence}}$ $\rho \vDash \mathcal{T}$

$\Downarrow$

$$\mathcal{T} \vDash_S \varphi \to \neg\neg\mathcal{T} \vdash_{CE} \varphi$$

Proof Outline

$\mathcal{T}$ consistent $\xrightarrow{\text{Model Existence}}$ $\rho \vDash \mathcal{T}$

$\Downarrow$

$$\text{stable}\,(\mathcal{T} \vdash_{CE} \varphi) \to \mathcal{T} \vDash_S \varphi \to \mathcal{T} \vdash_{CE} \varphi$$

Stability Necessity

For every standard model, we have

$$\text{stable}\,(\rho \vDash \varphi)$$

thus

$$\text{stable}\,(\mathcal{T} \vDash_S \varphi)$$

Stability Necessity

For every standard model, we have

$$\text{stable}\,(\rho \vDash \varphi)$$

thus

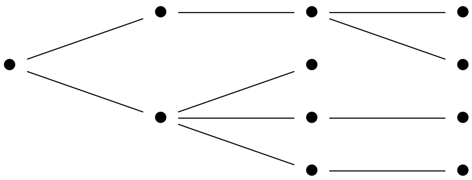
$$\text{stable}\,(\mathcal{T} \vDash_S \varphi)$$

As $\mathcal{T} \vDash_S \varphi \;\to\; \mathcal{T} \vdash_{CE} \varphi$ by soundness,

$$(\mathcal{T} \vDash_S \varphi \to \mathcal{T} \vdash_{CE} \varphi) \to \text{stable}\,(\mathcal{T} \vdash_{CE} \varphi)$$

Overview

Kripke Models



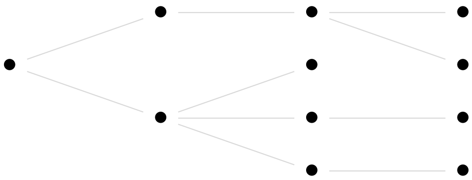
$$\mathcal{K} := (I, \mathcal{W}, \preccurlyeq, P_{\_}, \bot_{\_})$$

$$\forall\, u \preccurlyeq v.\ P_u\, t_1 \dots t_{|P|} \to P_v\, t_1 \dots t_{|P|}\ \wedge\ \bot_u \to \bot_v$$

Kripke Models

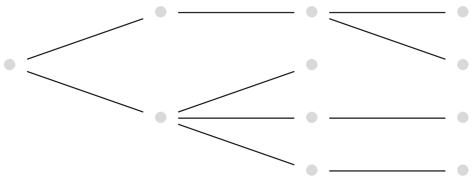
$$\mathcal{K} := (I, \mathcal{W}, \preccurlyeq, P_{\_}, \bot_{\_})$$

Kripke Models



$$\mathcal{K} := (I, \mathcal{W}, \preccurlyeq, P_{-}, \bot_{-})$$

Kripke Models



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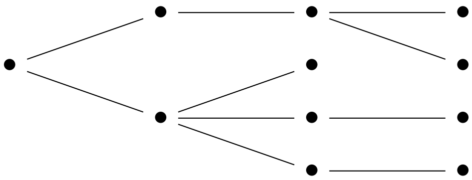
Kripke Models



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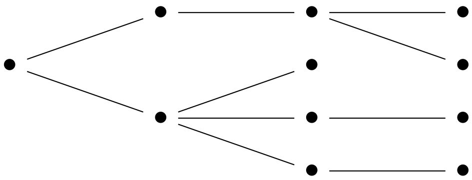
Kripke Models



$$\mathcal{K} := (I, \mathcal{W}, \preccurlyeq, P_{\_}, \bot_{\_})$$

$$\forall\, u \preccurlyeq v.\ P_u\, t_1 \ldots t_{|P|} \to P_v\, t_1 \ldots t_{|P|} \ \wedge\ \bot_u \to \bot_v$$

Kripke Semantics

Given a model $(I, \mathcal{W}, \preccurlyeq, P_{-}, \bot_{-})$ and a node $u : \mathcal{W}$, we define

$$\begin{aligned} \rho \Vdash_v \dot{\bot} &:= \bot_x^{\mathcal{K}} \\ \rho \Vdash_v P\, t_1 \ldots t_{|P|} &:= P_v^{\mathcal{K}}\, t_1^{\rho} \ldots t_{|P|}^{\rho} \\ \rho \Vdash_v \varphi \mathbin{\dot{\rightarrow}} \psi &:= \forall v \preccurlyeq w.\ \rho \Vdash_w \varphi \rightarrow \rho \Vdash_w \psi \\ \rho \Vdash_v \dot{\forall} \varphi &:= \forall d : D.\ d, \rho \Vdash_v \varphi \end{aligned}$$

$$A \Vdash_C \varphi := \forall \mathcal{K} u.\ C\, \mathcal{K} \rightarrow\ (\forall \psi \in A.\ \rho \Vdash_u \psi) \rightarrow \rho \Vdash_u \varphi$$

Kripke Model Notions

We define the constraint of **standard models** as

$$S\,\mathcal{K} := \forall u.\ \bot_u \to \bot$$

We define the constraint of **exploding models** as

$$E\,\mathcal{K} := \forall \rho, u, \varphi.\ \bot_u \to \rho \Vdash_u \varphi$$

We define any Kripke model to be a **minimal model**.

Normal Sequent Calculus

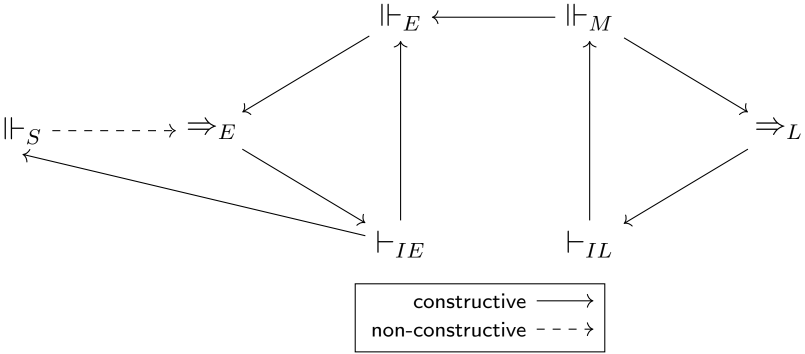
$$\textsc{Ax}\,\frac{}{\Gamma\,;\varphi \Rightarrow \varphi} \qquad \textsc{Ctx}\,\frac{\Gamma\,;\varphi \Rightarrow \psi \quad \varphi \in \Gamma}{\Gamma \Rightarrow \psi}$$

$$\text{IL}\,\frac{\Gamma \Rightarrow \varphi \quad \Gamma\,;\psi \Rightarrow \theta}{\Gamma\,;\varphi \dot{\to} \psi \Rightarrow \theta} \qquad \text{IR}\,\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \dot{\to} \psi}$$

$$\textsc{AllL}\,\frac{\Gamma\,;\varphi[t] \Rightarrow \psi}{\Gamma\,;\dot{\forall}\varphi \Rightarrow \psi} \qquad \textsc{AllR}\,\frac{\uparrow\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \dot{\forall}\varphi} \qquad \textsc{Exp}\,\frac{\Gamma \Rightarrow_E \dot{\bot}}{\Gamma \Rightarrow_E \varphi}$$

$$\mathcal{T} \Rightarrow \varphi := \exists \Gamma \subseteq \mathcal{T}.\ \Gamma \Rightarrow \varphi$$

Kripke Results



Overview

Dialogues

$$\underline{(P(x) \to Q(x)) \to P(x) \to P(x) \wedge Q(x)}$$

Dialogues

$$\underline{(P(x) \to Q(x)) \to P(x) \to P(x) \land Q(x)}$$

"Let's assume $P(x) \to Q(x)$." O: $P(x) \to Q(x)$

Dialogues

$$\underline{(P(x) \to Q(x)) \to P(x) \to P(x) \land Q(x)}$$

"Let's assume $P(x) \to Q(x)$." O: $P(x) \to Q(x)$

"Then $P(x) \to P(x) \land Q(x)$." P: $P(x) \to P(x) \land Q(x)$

Dialogues

$$\underline{(P(x) \to Q(x)) \to P(x) \to P(x) \land Q(x)}$$

"Let's assume $P(x) \to Q(x)$." O: $P(x) \to Q(x)$

"Then $P(x) \to P(x) \land Q(x)$." P: $P(x) \to P(x) \land Q(x)$

"Assuming $P(x)$, $P(x) \land Q(x)$ follows?" O: $A_{\to}\ P(x)$

Dialogues

$$\underline{(P(x) \to Q(x)) \to P(x) \to P(x) \land Q(x)}$$

"Let's assume $P(x) \to Q(x)$."	O: $P(x) \to Q(x)$
"Then $P(x) \to P(x) \land Q(x)$."	P: $P(x) \to P(x) \land Q(x)$
"Assuming $P(x)$, $P(x) \land Q(x)$ follows?"	O: $A_{\to}$ $P(x)$
"Yes."	P: $P(x) \land Q(x)$

Dialogues

$$\underline{(P(x) \to Q(x)) \to P(x) \to P(x) \wedge Q(x)}$$

"Let's assume $P(x) \to Q(x)$."	O: $P(x) \to Q(x)$
"Then $P(x) \to P(x) \wedge Q(x)$."	P: $P(x) \to P(x) \wedge Q(x)$
"Assuming $P(x)$, $P(x) \wedge Q(x)$ follows?"	O: $A_{\to}$ $P(x)$
"Yes."	P: $P(x) \wedge Q(x)$
"So $Q(x)$ holds?"	O: A_R

Dialogues

$$\underline{(P(x) \to Q(x)) \to P(x) \to P(x) \wedge Q(x)}$$

"Let's assume $P(x) \to Q(x)$." — O: $P(x) \to Q(x)$

"Then $P(x) \to P(x) \wedge Q(x)$." — P: $P(x) \to P(x) \wedge Q(x)$

"Assuming $P(x)$, $P(x) \wedge Q(x)$ follows?" — O: $A_{\to}\ P(x)$

"Yes." — P: $P(x) \wedge Q(x)$

"So $Q(x)$ holds?" — O: A_R

"As $P(x) \to Q(x)$, $Q(x)$ holds?" — P: $A_{\to}\ P(x)$

Dialogues

$$\underline{(P(x) \to Q(x)) \to P(x) \to P(x) \land Q(x)}$$

"Let's assume $P(x) \to Q(x)$."	O: $P(x) \to Q(x)$
"Then $P(x) \to P(x) \land Q(x)$."	P: $P(x) \to P(x) \land Q(x)$
"Assuming $P(x)$, $P(x) \land Q(x)$ follows?"	O: $A_{\to}\ P(x)$
"Yes."	P: $P(x) \land Q(x)$
"So $Q(x)$ holds?"	O: A_R
"As $P(x) \to Q(x)$, $Q(x)$ holds?"	P: $A_{\to}\ P(x)$
"Yes."	O: $Q(x)$

Dialogues

$$\underline{(P(x) \to Q(x)) \to P(x) \to P(x) \land Q(x)}$$

"Let's assume $P(x) \to Q(x)$."	O: $P(x) \to Q(x)$
"Then $P(x) \to P(x) \land Q(x)$."	P: $P(x) \to P(x) \land Q(x)$
"Assuming $P(x)$, $P(x) \land Q(x)$ follows?"	O: $A_{\to}$ $P(x)$
"Yes."	P: $P(x) \land Q(x)$
"So $Q(x)$ holds?"	O: A_R
"As $P(x) \to Q(x)$, $Q(x)$ holds?"	P: $A_{\to}$ $P(x)$
"Yes."	O: $Q(x)$
"Then $Q(x)$ holds."	P: $Q(x)$

Attacks & Defenses

$$\rhd : \mathfrak{F} \to \mathcal{A} \to \mathcal{O}(\mathfrak{F}) \to \mathbb{P}$$
$$\mathcal{D}_{.} : \mathcal{A} \to 2^{\mathfrak{F}}$$

Attacks	$\mathcal{D}_a$
$A_\bot \rhd \dot{\bot}$	—
$A_\to \,|\, \ulcorner\varphi\urcorner \rhd \varphi \dot{\to} \psi$	$\{\psi\}$
$A_\vee \rhd \varphi \dot{\vee} \psi$	$\{\varphi, \psi\}$
$A_L \rhd \varphi \dot{\wedge} \psi$	$\{\varphi\}$
$A_R \rhd \varphi \dot{\wedge} \psi$	$\{\psi\}$
$A_t \rhd \dot{\forall}\varphi$	$\{\varphi[t]\}$
$A_\exists \rhd \dot{\exists}\varphi$	$\{\varphi[t] \mid t : \mathfrak{T}\}$

$$a \rhd \varphi := a \,|\, \emptyset \rhd \varphi$$

Sequent Calculus LJD

$$\Rightarrow_D \colon \mathcal{L}(\mathfrak{F}) \to 2^{\mathfrak{F}} \to \mathbb{P}$$

$$\text{L}\,\frac{\begin{array}{c}\varphi \in \Gamma \quad a \,|\, \psi \rhd \varphi \\ \text{justified } \Gamma\, \psi \quad \forall \sigma \in \mathcal{D}_a.\ \Gamma, \sigma \Rightarrow_D \Delta \quad \forall a' \,|\, \tau \rhd \psi.\ \Gamma, \tau \Rightarrow_D \mathcal{D}_{a'}\end{array}}{\Gamma \Rightarrow_D \Delta}$$

$$\text{R}\,\frac{\varphi \in \Delta \quad \text{justified } \Gamma\, \varphi \quad \forall a \,|\, \psi \rhd \varphi.\ \Gamma, \psi \Rightarrow_D \mathcal{D}_a}{\Gamma \Rightarrow_D \Delta}$$

Full Intuitionistic Sequent Calculus I

$$\text{Ax}\,\frac{}{\Gamma, \varphi \Rightarrow_J \varphi} \qquad \textsc{Contr}\,\frac{\Gamma, \varphi, \varphi \Rightarrow_J \psi}{\Gamma, \varphi \Rightarrow_J \psi}$$

$$\textsc{Weak}\,\frac{\Gamma \Rightarrow_J \psi}{\Gamma, \varphi \Rightarrow_J \psi} \qquad \textsc{Perm}\,\frac{\Gamma, \psi, \varphi, \Gamma' \Rightarrow_J \theta}{\Gamma, \varphi, \psi, \Gamma' \Rightarrow_J \theta}$$

$$\textsc{Exp}\,\frac{\Gamma \Rightarrow_J \dot{\bot}}{\Gamma \Rightarrow_J \varphi} \qquad \text{TR}\,\frac{}{\Gamma \Rightarrow_J \dot{\top}}$$

$$\text{IL}\,\frac{\Gamma \Rightarrow_J \varphi \quad \Gamma, \psi \Rightarrow_J \theta}{\Gamma, \varphi \dot{\to} \psi \Rightarrow_J \theta} \qquad \text{IR}\,\frac{\Gamma, \varphi \Rightarrow_J \psi}{\Gamma \Rightarrow_J \varphi \dot{\to} \psi}$$

Full Intuitionistic Sequent Calculus II

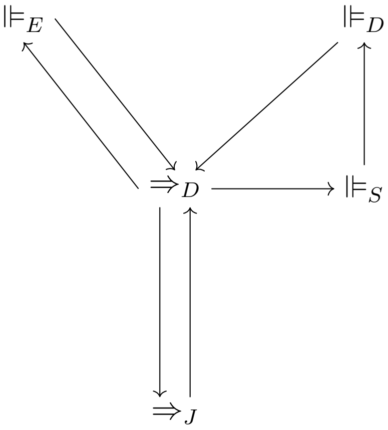
$$\text{AL}\,\frac{\Gamma, \varphi, \psi \Rightarrow_J \theta}{\Gamma, \varphi \mathbin{\dot\wedge} \psi \Rightarrow_J \theta} \qquad \text{AR}\,\frac{\Gamma \Rightarrow_J \varphi \quad \Gamma \Rightarrow_J \psi}{\Gamma \Rightarrow_J \varphi \mathbin{\dot\wedge} \psi}$$

$$\text{OL}\,\frac{\Gamma, \varphi \Rightarrow_J \theta \quad \Gamma, \psi \Rightarrow_J \theta}{\Gamma, \varphi \mathbin{\dot\vee} \psi \Rightarrow_J \theta} \qquad \text{OR}_{\text{L}}\,\frac{\Gamma \Rightarrow_J \varphi}{\Gamma \Rightarrow_J \varphi \mathbin{\dot\vee} \psi}$$

$$\text{OR}_{\text{R}}\,\frac{\Gamma \Rightarrow_J \psi}{\Gamma \Rightarrow_J \varphi \mathbin{\dot\vee} \psi} \qquad \text{AllL}\,\frac{\Gamma, \varphi[t] \Rightarrow_J \psi}{\Gamma, \dot\forall\varphi \Rightarrow_J \psi}$$

$$\text{AllR}\,\frac{\uparrow\Gamma \Rightarrow_J \varphi}{\Gamma \Rightarrow_J \dot\forall\varphi} \qquad \text{ExL}\,\frac{\uparrow\Gamma, \varphi \Rightarrow_J \uparrow\psi}{\Gamma, \dot\exists\varphi \Rightarrow_J \psi} \qquad \text{ExR}\,\frac{\Gamma \Rightarrow_J \varphi[t]}{\Gamma \Rightarrow_J \dot\exists\varphi}$$

Dialogue Results



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