Overview of Löwenheim-Skolem Theorem

Main results under constructive logic

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1 Löwenheim-Skolem Theorem

1.1 Definition

Definition 1 (Elementary Embedding)**.** A injective homomorphism h from models $\mathcal{N}$ to $\mathcal{M}$ is elementary if all formulas $\varphi : \mathbb{F}$ are preserved under this homomorphism, formally

$$\mathcal{N} \vDash_\rho \varphi \iff \mathcal{M} \vDash_{h \circ \rho} \varphi,$$

where ρ is an environment assigning values to free variables.

If an elementary embedding from model $\mathcal{N}$ to $\mathcal{M}$ exists, $\mathcal{N}$ is an is an elementary submodel of $\mathcal{M}$, denoted as:

$$\mathcal{N} \preceq_h \mathcal{M} := h \text{ is an elementary homomorphism from } \mathcal{N} \text{ to } \mathcal{M}.$$

Definition 2 (Henkin witness)**.** For any formula φ under an environment ρ, Henkin witness w defined as:

$$\mathcal{M} \vDash_\rho \varphi[w] \to \mathcal{M} \vDash_\rho \dot\forall \varphi \quad \text{or} \quad \mathcal{M} \vDash_\rho \dot\exists \varphi \to \mathcal{M} \vDash_\rho \varphi[w].$$

Definition 3 (The witness property)**.** A model $\mathcal{M}$ satisfies the witness property if the Henkin witness of any formula $\varphi : \mathbb{F}$ can be evaluated by a closed term t, formally:

$$\exists t : \mathbb{T}^c.\ \mathcal{M} \vDash \varphi[t] \to \mathcal{M} \vDash \dot\forall \varphi.$$

Definition 4 (Henkin Environment)**.** An environment $\rho : \mathbb{N} \to \mathcal{M}$ is called Henkin environment if for all formulas $\varphi : \mathbb{F}$:

$$(\forall n : \mathbb{N}.\ \mathcal{M} \vDash_\rho \varphi[x_n]) \to \mathcal{M} \vDash_\rho \dot\forall \varphi$$
$$\mathcal{M} \vDash_\rho \dot\exists \varphi \to (\exists n : \mathbb{N}.\ \mathcal{M} \vDash_\rho \varphi[x_n]).$$

Definition 5 (Löwenheim-Skolem Theorem (LS))**.** For any model $\mathcal{M}$, there exists an elementary embeddding h to a countable submodel $\mathcal{N}$.

$$\mathcal{N} \preceq_h \mathcal{M}$$

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1.2 Result

Theorem 1 (Löwenheim-Skolem Theorem I)**.** For any classical and nonempty model $\mathcal{M}$ with a countable signature, there is a countable syntactic model $\mathcal{N}$ such that any closed formula $\varphi : \mathbb{F}^c$ satisfies

$$\mathcal{M} \vDash \varphi \iff \mathcal{N} \vDash \varphi.$$

Theorem 2 (Löwenheim-Skolem Theorem II)**.** For any model $\mathcal{M}$ with a function $i : \mathbb{N} \to \mathcal{M}$, if $\mathcal{M}$ satifies the witness property, then there is a elementary embedding from the syntactic model $\mathcal{N}_i$ to $\mathcal{M}$:

$$\mathcal{N}_i \preceq_{\hat{i}} \mathcal{M}$$

Theorem 3 (Löwenheim-Skolem Theorem III)**.** For any model $\mathcal{M}$, if the environment ι is Henkin, then

$$\mathcal{N}_\iota \preceq_{\hat{\iota}} \mathcal{M},$$

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2 Axiom of Dependent Choice

2.1 Definition

Axiom 1 (Blurred Drinker Paradox (BDP))**.**

$$\forall A.\, \forall P : A \to \mathfrak{P}.\, \exists b : \mathbb{N} \to A.\, (\forall n.\, P\,(b\,n)) \to \forall x.\, P\,x.$$

Axiom 2 (Dual form of Blurred Drinker Paradox (BDP′))**.**

$$\forall A.\, \forall P : A \to \mathfrak{P}.\, \exists b : \mathbb{N} \to A.\, (\exists x.\, P\,x) \to \exists n.\, P(b\,n).$$

Axiom 3 (Countable Choice (CC))**.** For any total relation $R : \mathbb{N} \to A \to \mathfrak{P}$ over a countable set, there is a function $f : \mathbb{N} \to A$, s.t.

$$\forall n.\, R\,n\,(f\,n).$$

Axiom 4 (Blurred Countable Choice (BCC))**.** For any total relation $R : \mathbb{N} \to A \to \mathfrak{P}$ over a countable set, there is a function $f : \mathbb{N} \to A$, s.t.

$$\forall n.\, \exists m\; R\,n\,(f\,m).$$

Axiom 5 (Dependent Choice (DC))**.** For any total relation $R : A \to A \to \mathfrak{P}$,

$$\exists f : \mathbb{N} \to A. \forall n.\, R\,(f\,n)\,(f\,(n+1)).$$

Axiom 6 (Blurred Dependent Choice (BDC))**.** For any total ternary relation $R : A \to A \to A \to \mathfrak{P}$,

$$\exists f : \mathbb{N} \to A. \forall n\,m.\, \exists k.\, R\,(f\,n)\,(f\,m)\,(f\,k).$$

Axiom 7 (Omniscient Blurred Dependent Choice (OBDC))**.** For any ternary relation $R : A \to A \to A \to \mathfrak{P}$,

$$\exists f : \mathbb{N} \to A.\, (\forall x\,y.\, \exists z.\, R\,x\,y\,z) \iff \forall n\,m.\, \exists k.\, R\,(f\,n)\,(f\,m)\,(f\,k).$$

Axiom 8 (Directed Dependent Choice (DDC))**.** For any directed and transitive binary relation $R : A \to A \to \mathfrak{P}$,

$$\exists f : \mathbb{N} \to A. \forall n\ m.\ \exists k.\ R\ (f\ n)\ (f\ k) \wedge R\ (f\ m)\ (f\ k).$$

Remark.

$$\begin{aligned} \mathsf{OBDC} &\Rightarrow \mathsf{BDC} \Rightarrow \mathsf{DDC} \\ \mathsf{OBDC} &\Rightarrow \mathsf{BDP} \\ \mathsf{OBDC} &\Rightarrow \mathsf{BDP}' \\ \mathsf{BDC} &\Rightarrow \mathsf{BCC} \\ \mathsf{DC} &\Rightarrow \mathsf{DDC} + \mathsf{BCC} \iff \mathsf{BDC} \\ \mathsf{BDC} + \mathsf{AC}_{\mathbb{N},\mathbb{N}} &\Rightarrow \mathsf{DC} \end{aligned}$$

We can also define LBDC to be a relation R over list of any set A.

Axiom 9 (List Blurred Dependent Choice (LBDC))**.** For any total relation $R : \mathcal{L}(A) \to A \to \mathfrak{P}$,

$$\exists f : \mathbb{N} \to A. \forall\, l : \mathcal{L}(A).\ \exists m.\ R\ (\hat{f}\ l)\ (f\ m).$$

There is:

$$\begin{aligned} \mathsf{LBDC} &\Rightarrow \mathsf{BDC} \\ \mathsf{LS} &\Rightarrow \mathsf{LBDC} \end{aligned}$$

But now it's not very important. Additional, there are following facts about DC in constructive logic:

$$\begin{aligned} \mathsf{LS} \wedge R \text{ is decidable} &\to \mathsf{DC} \text{ on } R \\ \mathsf{LS} \wedge R \text{ is definite} &\to \mathsf{DC}_{\mathsf{prop}} \text{ on } R \end{aligned}$$

2.2 Result

Theorem 4 (Equivalent to LS I)**.**

$$\mathsf{DDC} + \mathsf{BCC} + \mathsf{BDP} + \mathsf{BDP}' \iff \mathsf{LS}$$

Theorem 5 (Equivalent to LS II)**.**

$$\mathsf{BDC} + \mathsf{BDP} + \mathsf{BDP}' \iff \mathsf{LS}$$

Theorem 6 (Equivalent to LS III)**.**

$$\mathsf{OBDC} \iff \mathsf{LS}$$

Theorem 7 (Equivalent to LS IV)**.**

$$\mathsf{DC} + \mathsf{BDP} + \mathsf{BDP}' + \mathsf{AC}_{\mathbb{N},\mathbb{N}} \iff \mathsf{LS} + \mathsf{AC}_{\mathbb{N},\mathbb{N}}$$

2.3 About BDP and BDP′

Definition 6 (DP_A^B and DP'^B_A)**.** General blurred form of Drinker Paradox DP_A^B over types A and B is defined by:

$$\forall R : B \to \mathfrak{P}.\ \exists f : A \to B.\ (\forall a.\ P\ (f\ a)) \to \forall x.\ P\ x$$

Let $\mathsf{DP}_A := \forall B.\ \mathsf{DP}_A^B$, then $\mathsf{DP} = \mathsf{DP}_{\mathbb{N}}$.

Also, the dual form DP'^B_A is defined as follow:

$$\forall R : B \to \mathfrak{P}.\ \exists f : A \to B.\ (\exists x.\ P\ x) \to \exists a.\ P\ (f\ a).$$

Fact 8.

$$\mathsf{BDP}_A + \mathsf{BDP}_{\mathbb{I}}^A \iff \mathsf{BDP} \iff \mathsf{LEM}$$
$$\mathsf{BDP}'_A + \mathsf{BDP}'^A_{\mathbb{I}} \iff \mathsf{BDP}' \iff \mathsf{LEM},$$

Definition 7 (Limited Principle of Omniscience (LPO))**.**

$$\forall f : \mathbb{N} \to \mathbb{B}.\ (\forall x.\ fx = \mathsf{false}) \lor (\exists x.\ fx = \mathsf{true})$$

Definition 8 (Independence of Premise (IP))**.**

$$\forall (P : A \to \mathfrak{P})(Q : \mathfrak{P}).\ A \to (Q \to \exists x.\ Px) \to \exists x.\ Q \to P\ x.$$

Fact 9.

$$\mathsf{LEM} \iff \mathsf{BDP}_{\mathbb{N}} + \mathsf{BDP}_{\mathbb{I}}^{\mathbb{N}} \iff \mathsf{BDP}_{\mathbb{N}} + \mathsf{LPO} \iff \mathsf{IP}$$
$$\mathsf{BDP}_{\mathbb{I}}^{\mathbb{N}} \Rightarrow \mathsf{LPO}$$

3 General Remarks

Without considering any axioms, we can either strengthen the requirements or weaken the results under constructive logic. In summary we have:

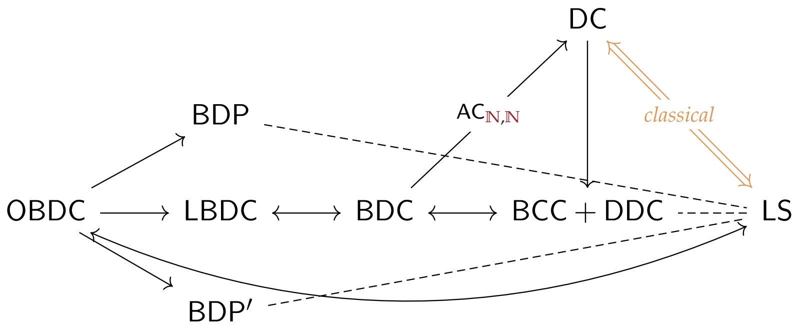
$$\mathcal{M} \text{ is classical} \Rightarrow \text{weak form of } \mathsf{LS}$$
$$\mathcal{M} \text{ satisfies the witness property} \Rightarrow \mathsf{LS}$$

We now consider the constructive logic equipped with the countable choice axiom (which can be weakened to $\mathsf{AC}_{\mathbb{N},\mathbb{N}}$):

$$\mathsf{DC} + \mathsf{BDP} + \mathsf{BDP}' \iff \mathsf{LS}$$

Removing the countable choice axiom gives:

$$\mathsf{BDC} + \mathsf{BDP} + \mathsf{BDP}' \iff \mathsf{LS}$$
$$\mathsf{DDC} + \mathsf{BCC} + \mathsf{BDP} + \mathsf{BDP}' \iff \mathsf{LS}$$
$$\mathsf{OBDC} \iff \mathsf{LS}$$



All proofs except BDC → LBDC are verified in `Coq`. (The 3 dashed lines mean that they together have the strength of LS)

Remark (Upward LS). With respect to the upward LS, since we do not require first-order theories to contain equivalent symbols, this means that no quotient of the model is required. Thus we have a trivial upward part of the Löwenheim-Skolem theorem, i.e., copying some element makes the model have an arbitrarily large cardinality. (The model quotient by equivalence relation would cause these elements to collapse to a single element).

One thing is that in addition to our results, the approach of our proof is also a little bit novel (as far as I know). We avoid any modification of the model and signature by iterating over the domain of the environment and mapping the values to free variables, thus directly obtaining an elementary syntactic model, which also restricts us to discussing only on countable models.

(Comment: There are similar ways of proving this, such as in the Wiki entry for the LS theorem, which does not explicitly get an environment like this, but also gets a countable set via the Skolem function (which is non-constructive), which is then added to the signature at the end.)

4 Conclusion

Finally, why does constructive math tell us more? As the figure shows, under the ground, i.e., in a world without the LEM, DC and LS are not equivalent.

Figure 1: LS: Under the ground