THE DOWNWARD LÖWENHEIM-SKOLEM THEOREM AND THE BLURRED DRINKER PARADOX

HAOYI ZENG

1. Abstract

The Downward Löwenheim-Skolem theorem (DLS) is a fundamental meta-theorem of first-order logic, stating that any infinite model (over a countable signature) has a countably infinte elementary submodel. It is a well-known result in reverse mathematics that DLS is equivalent to dependent choice (DC) over classical foundations such as ZF set theory. In this talk we reexamine this equivalence from the perspective of constructive reverse mathematics over dependent type theory, allowing for a finer analysis of the necessary logical principles.

Indeed, complementing the connection to DC, we identify a weak classical principle we call the blurred drinker paradox (BDP) that necessarily contributes to DLS. Concretely, we describe a new proof of DLS highlighting the use of DC and BDP otimised for formalisation in type theory and mechanisation in a proof assistant. This is prepared by preliminary, fully-constructive variants of DLS with stronger assumptions or weaker onclusions. We end with a few general remarks on BDP and related logical principles.

2. Löwenheim-Skolem Theorem: First glance

In this report, we use the calculus of inductive constructions (CIC) [**?**] as the mathematical foundation for formalizing first-order logic [**?**]. We represent terms $\mathbb{T}$ and formulas $\mathbb{F}$ of first-order logic as inductive types over a fixed signature $(\mathcal{F}_\Sigma, \mathcal{P}_\Sigma)$. Especially, $\mathbb{F}^c$ and $\mathbb{T}^c$ represent all closed terms and formulas respectively.

The logical connectives with dots indicate the object logic, otherwise the meta-logic. For instance, in most of the content of this report, we discuss the theorems in the negative fragment syntax defined over $(\dot\forall, \dot\rightarrow, \dot\bot)$. The syntax that includes all the logical connectives $(\dot\forall, \dot\exists, \dot\wedge, \dot\vee, \dot\rightarrow, \dot\bot)$ is called full syntax.

Based on the de Bruijn style of free variable binding, an assignment function, or environment is defined as a function from $\mathbb{N}$ to the domain of model that semantically assigns all free variables to elements in a model domain. Additionally, a substitution $\sigma : \mathbb{N} \to \mathbb{T}$ is defined to syntactically replace the free variables in a formula φ with corresponding terms, denoted as $\varphi[\sigma] : \mathbb{F}$.

Definition 1 (Syntactic Model I)**.** For any theories $\mathcal{T}$ over signature $(\mathcal{F}_\Sigma, \mathcal{P}_\Sigma)$, the syntactic model $\mathcal{N}_\mathcal{T}$ (also called Henkin model) is defined as a model over type $\mathbb{T}$ of terms by setting the semantics of functions $f : \mathcal{F}_\Sigma$ and predicates $P : \mathcal{P}_\Sigma$ as:

$$f^\mathcal{T} t := f\ t \qquad P^\mathcal{T} t := Pt \in \mathcal{T}.$$

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We start with the Henkin-style completeness proof based on [?]. The main result is the extension of a consistent theory $\mathcal{T}$ to $\Omega(\mathcal{T})$ such that $\Omega(\mathcal{T})$ is consistent, deductively closed, respects implication, and respects universal quantification, and gives rise to equivalent syntactic models $\mathcal{N}_{\Omega(\mathcal{T})}$ over the domain $\mathbb{T}$ of terms. Since the syntactic model $\mathcal{N}_{\Omega(\mathcal{T})}$ is countable, this suggests a direct method of proving the DLS theorem.

The DLS theorem, specifically the downward part of the DLS theorem at countable cardinality, states that for any model with a countable signature, there exists a countable submodel.

Based on the existence of the syntactic model, the DLS theorem can be derived as a corollary of the completeness theorem.

Theorem 1 (Löwenheim-Skolem Theorem I)**.** For any classical and nonempty model $\mathcal{M}$ with a countable signature, there is a countable syntactic model $\mathcal{N}$ such that any closed formula $\varphi : \mathbb{F}^c$ satisfies

$$\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi.$$

Proof. Since the model $\mathcal{M}$ is classical and nonempty, the theory of this model $\mathrm{Th}(\mathcal{M})$ is consistent via the soundness theorem. As in the proof in the completeness theorem, the syntactic model $\mathcal{N}_{\mathcal{T}'}$ exists, where $\mathcal{T}' := \Omega(\mathrm{Th}(\mathcal{T}))$. The correctness of the syntactic model shows that

$$\forall \varphi \colon \mathbb{F}^c.\ \varphi \in \mathcal{T}' \iff \mathcal{N} \models \varphi.$$

Since $\mathrm{Th}(\mathcal{M})$ is defined as $\mathcal{M} \models \varphi$ for all colosed formulas $\varphi : \mathbb{F}^c$, the only gap is that

$$\varphi \in \mathrm{Th}(\mathcal{M}) \iff \varphi \in \mathcal{T}'.$$

The left direction is shown by the classicality of this model as well as the consistency of $\mathrm{Th}(\mathcal{M})$, and the right direction is the property of the Henkin model. □

This version of the DLS theorem is preferred in most mechanized proofs (e.g., Isabelle/HOL [?], and Mizar [?]), as it can be directly derived as a corollary of the completeness theorem without requiring any axiom of choice.

3. Löwenheim-Skolem Theorem: the Witness Property

On the other hand, the widely known DLS theorem [?] that is obtained by assuming the Axiom of Choice (AC)[1] is stated as follows: for any model over a countable signature, there exists a countable **elementary submodel**. This stronger version of the theorem demonstrates that the countable submodel is elementary, meaning that for any open formula, not just closed ones, its truth value can be preserved between two models when the environment is restricted to the submodel domain.

Definition 2 (Elementary Embedding)**.** A injective homomorphism h from models $\mathcal{N}$ to $\mathcal{M}$ is elementary if all formulas $\varphi : \mathbb{F}$ are preserved under this homomorphism, formally

$$\mathcal{N} \models_\rho \varphi \iff \mathcal{M} \models_{h \circ \rho} \varphi,$$

where ρ is an environment assigning values to free variables.

[1]More precisely, many people in the past also observed that DLS is equivalent to DC, for example, George Boolos observed in [?].

If an elementary embedding from model $\mathcal{N}$ to $\mathcal{M}$ exists, $\mathcal{N}$ is an is an elementary submodel of $\mathcal{M}$, denoted as:

$$\mathcal{N} \preceq_h \mathcal{M} := h \text{ is an elementary homomorphism from } \mathcal{N} \text{ to } \mathcal{M}.$$

The first proof of the DLS theorem does not provide a direct solution here since not all syntactic models are elementary submodels. The most challenging case is when dealing with quantifiers, where proving a formula with a quantifier (either universal or existential) often requires the existence of a specific witness.

Therefore, we define the *Henkin witness* w for a formula φ under an environment ρ is

$$\mathcal{M} \vDash_{w;\rho} \varphi \to \mathcal{M} \vDash_\rho \dot{\forall}\varphi \quad \text{or} \quad \mathcal{M} \vDash_\rho \dot{\exists}\varphi \to \mathcal{M} \vDash_{w;\rho} \varphi.$$

The reason why the Henkin model is not an elementary submodel lies in the fact that some Henkin witnesses are not captured, even though we can assign the Henkin witness c_φ to each formula φ during the Henkinization process. Once c_φ is added to the domain of the model, new Henkin witnesses may be generated because the structures of the model has changed.

The one-and-done premise, therefore, is *the witness property*, stating that if all the witnesses already exist and are captured by some closed term. (as long as the term is invariant under any environments, it is sufficient, but for convenience, we use the closed term.)

Definition 3 (The witness property)**.** A model $\mathcal{M}$ satisfies the witness property if the Henkin witness of any formula $\varphi : \mathbb{F}$ can be evaluated by a closed term t, formally:

$$\exists t : \mathbb{T}^c.\ \mathcal{M} \vDash \varphi[t] \to \mathcal{M} \vDash \dot{\forall}\varphi.$$

If all witnesses can be expressed in a closed term, then the free variables in the domain of the syntactic model become redundant, as they were only added as witnesses in the Henkinization. However, the DLS theorem specifies that the submodel contains at most countably many elements, then we can define a generalized syntactic model as follows:

Definition 4 (Syntactic Model II)**.** For any model $\mathcal{M}$ over signature $(\mathcal{F}_\Sigma, \mathcal{P}_\Sigma)$, with a function $i : \mathbb{N} \to \mathcal{M}$ (considered to select a countable subset of $\mathcal{M}$), the syntactic model $\mathcal{N}_i$ is defined as a model over the term types $\mathbb{T}$ by setting the semantics of functions $f : \mathcal{F}_\Sigma$ and predicates $P : \mathcal{P}_\Sigma$ as:

$$f^{\mathcal{N}} t := f\ t \qquad P^{\mathcal{N}} t := M \vDash_i P\ t,$$

The syntactic model $\mathcal{N}_i$ can be regarded as the submodel generated by the range of i, that is, $i(\mathbb{N})$, up to isomorphism. This observation is essential to the upcomming discussion of submodels without altering the original model, but rather by constructing a new model. (see section 5)

Once again, the DLS theorem can be proven under the witness property.

Theorem 2 (Löwenheim-Skolem Theorem II)**.** For any model $\mathcal{M}$ with a function $i : \mathbb{N} \to \mathcal{M}$, if $\mathcal{M}$ satifies the witness property, then the domain of the syntactic model $\mathcal{N}_i$ contains the range of i through the elementary embedding h, such that

$$\mathcal{N}_i \preceq_h \mathcal{M},$$

where $h : \mathbb{T} \to \mathcal{M}$ is the evaluation of a term under the environment i.

Proof. The interpretation of the Henkin model under i would satisfy the following condition for all formulas φ:

$$\exists t.\ \mathcal{M} \models_h \varphi[t] \dot{\rightarrow} \dot{\forall}\varphi$$

We now proceed to verify that h constitutes an elementary embedding from $\mathcal{N}$ to $\mathcal{M}$ by induction on the formula φ. The only non-trivial case is as follows:

$$\forall \phi\ \rho.\ \mathcal{N}_i \models_\rho \dot{\forall}\phi \rightarrow \mathcal{M} \models_{h \circ \rho} \dot{\forall}\phi.$$

To establish this, we apply the witness property and demonstrate that:

$$\mathcal{M} \models_{h \circ (t;\rho)} \phi \rightarrow \mathcal{M} \models_{h \circ \rho} \phi[t],$$

which is derived from the fact that the term $t : \mathbb{T}$ evaluated under $h \circ \rho$ is equal to $h(t)$, given that t is closed. □

Thus far, we have demonstrated how DLS can be proved from stronger premises or for weaker conclusions, but we have not yet presented the standard proof of DLS.

In the next section, we will discuss when a model fails to satisfy the witness property, there may still exist a fixed environment ι such that $\mathcal{N}_\iota$ is an elementary submodel. As we shall see later, it is possible to construct this environment using DC.

4. Löwenheim-Skolem Theorem: Henkin Environment

The aforementioned proof can serve as inspiration for finding an appropriate environment ι such that $\mathcal{N}_\iota$ is an elementary submodel. By mapping to the term on top of the free variables, $\mathcal{N}_\iota$ can specify any subset, or even several subsets of $\mathcal{M}$ as part of its domain. It is conjectured that it is enough for ι to include all possible witnesses. In other words, the environment ι should contain all possible Henkin witnesses for all formulas evaluated under ι. This specification is defined as follows.

Definition 5 (Henkin Environment)**.** An environment $\rho : \mathbb{N} \rightarrow \mathcal{M}$ is called Henkin environment if for all formulas $\varphi : \mathbb{F}$:

$$(\forall n : \mathbb{N}.\ \mathcal{M} \models_{\rho(n);\rho} \varphi) \rightarrow \mathcal{M} \models_\rho \dot{\forall}\varphi.$$

This definition is related to the witness property, in the case that all elements corresponding to this environment are added to the model as constants, then the model satisfies the witness property.

Theorem 3 (Löwenheim-Skolem Theorem III)**.** For any model $\mathcal{M}$, if the environment ι is Henkin, then

$$\mathcal{N}_\iota \preceq_h \mathcal{M},$$

where $h : \mathbb{T} \rightarrow \mathcal{M}$ is the evaluation of a term under the environment ι.

Proof. Similar to previous proofs, through induction, only the case with quantifier is non-trivial.

$$\forall \phi\ \rho.\ \mathcal{N}_\iota \models_\rho \dot{\forall}\phi \rightarrow \mathcal{M} \models_{h \circ \rho} \dot{\forall}\phi.$$

Now, the standard induction is unable to solve this proof, what we need is induction under any substitution, which means that any substitution in the inductive hypothesis for ϕ can be used. In other words, our induction hypothesis is

$$\mathcal{N}_\iota \models_\rho \phi[\sigma] \iff \mathcal{M} \models_{h \circ \rho} \phi[\sigma]$$

First, change the goal by the property of substitution:

$$\mathcal{M} \vDash_{h \circ \rho} \dot{\forall} \phi \iff \mathcal{M} \vDash_{\iota} (\dot{\forall} \phi)[\rho],$$

Since the environment is Henkin, assuming the formula $\phi[\uparrow \rho]$, where $\uparrow$ increases the number of the free variable from n to $(n+1)$ s.t. $\dot{\forall}\phi[\uparrow \rho] = (\dot{\forall}\phi)[\rho]$, we can obtain:

$$(\forall n : \mathbb{N}.\ \mathcal{M} \vDash_{f(n);f} \phi[\uparrow \rho]) \to \mathcal{M} \vDash_{\rho} (\dot{\forall}\phi)[\rho]$$

This proof is completed by applying the induction hypothesis on the $\dot{\forall}\phi$ of the term model with n,

$$\mathcal{N} \vDash_{n;\rho} \phi \iff \mathcal{M} \vDash_{f(n);f} \phi[\uparrow \rho].$$

□

The premise that DLS is established is evident, to achieve a complete version of DLS, the construction of a Henkin environment is necessary, which can be obtained through some choice principles (see section 5).

The Henkin environment implies the DLS theorem is closely connected with to the Tarski-Vaught test:

Lemma 1 (Tarski-Vaught Test)**.** Any submodel $\mathcal{N}$ of $\mathcal{M}$ is an elementary submodel if and only if for any formula $\varphi : \mathbb{F}$ and environment ρ over $\mathcal{N}$,

$$(\exists m : \mathcal{M}.\ \mathcal{M} \vDash_{m;\rho} \varphi) \to \exists n.\ \mathcal{N} \vDash_{n;\rho} \varphi.$$

The difference is that in the Tarski-Vaught test, we consider a submodel that is restricted to a subset of the original model and contains all Henkin witnesses.

However, by using the definition of a Henkin environment, we avoid constructing a submodel and instead increase the environment to restrict the choice of elements. This is more important for the subsequent proof because we only need to recursively define an environment instead of different submodel objects.

5. Construction of Henkin Environment

To construct a Henkin environment, we begin with an arbitrary environment and apply dependent choice (DC) and a blurred form BDP of the drinker's paradox (DP) to obtain a chain of environments accumulating Henkin witnesses. With this chain in hand, we can then construct a fixed point that is Henkin.

DC is a principle in mathematical logic that is strictly weaker than the Axiom of Choice. It states that for any total binary relation $R : A \to A \to \mathfrak{P}$, there exists an index function $F : \mathbb{N} \to A$ such that $R(F\ n, F(n+1))$ for any natural number n, and the root of F can be chosen freely. More specifically, DC can be formalized as follows:

Axiom 1 (Functional Dependent Choice)**.**

$$\forall R : A \to A \to \mathfrak{P}.\ (\forall x.\ \exists y.\ R\ x\ y) \to \exists f : \mathbb{N} \to A. \forall n.\ R\ (f\ n)\ (f\ (n+1)).$$

In other words, Dependent Choice allows us to make a sequence of choices where each choice depends on the previous ones, as long as there is always at least one possible choice available at each step (expressed as totality).

The DP is a classical logical paradox that can be derived from the Law of Excluded Middle (LEM). It demonstrates that for the proposition $\forall x.Px$ to hold, it is always possible to find a witness w such that $Pw \to \forall x.Px$.

However, since this assumption is already too strong for the proof of the DLS theorem, it is not possible to infer the DP from DLS. In principle, it is sufficient for DLS that this w is hidden in a countable set. Therefore, the blurred form of the Drinking Paradox BDP can be defined.

Axiom 2 (Blurred Drinker Paradox)**.**

$$\forall P : A \to \mathfrak{P}.\ \exists I : \mathbb{N} \to A.\ (\forall n.\ P\ (I\ n)) \to \forall x.\ P\ x.$$

This is a certain kind of omniscience in constructive proofs since it is possible to predict in advance the existence of a countable set containing the Henkin witness.

One more step, the Axiom of Countable Choice(AC_ω) will be used, however, as will be shown in the reversed direction (see section 6), AC_ω is too strong, therefore, we consider a blurred form of BAC_ω, said BAC_ω.

Definition 6 (Blurred Countable Choice)**.** For any total relation $R : \mathbb{N} \to A \to \mathfrak{P}$ over a countable set, there is a function $f : \mathbb{N} \to A$, s.t.

$$\forall n.\ \exists m.\ R\ n\ (f\ m).$$

Since the BAC_ω can be derived from DC, it is not strictly necessary, and it serves the purpose of selecting countable environments that contain a Henkin witness.

Lemma 2 (Totality)**.** Based on the definition of the Henkin environment, an aid relation $I : (\mathbb{N} \to \mathcal{M}) \to (\mathbb{N} \to \mathcal{M}) \to \mathfrak{P}$ defined by:

$$I\ \rho\ \rho_s := \forall \varphi.\ (\forall m.\ \mathcal{M} \vDash_{(\rho_S\ m).:\rho} \varphi) \to \mathcal{M} \vDash_\rho \dot{\forall}\varphi\ \wedge\ \rho\ \subseteq \rho_s,$$

The notation $\rho \subseteq \rho_s$ means that the image of ρ is contained in ρ_s as defined by $\forall x.\ \exists y.\ \rho\ x = \rho_s\ y$.

Assuming the axioms BDP and DC, the relation I is total.

Proof. Given any environment ρ, the existence of a witness function $i : \mathbb{N} \to \mathcal{M}$ for each formula φ is implied by BDP, such that:

$$(\forall n.\ \mathcal{M} \vDash_{i(n);\rho} \varphi) \to \mathcal{M} \vDash_\rho \dot{\forall}\varphi.$$

Since the signature is countable, there exists a corresponding witness function for each formula indexed by I.

We can then obtain a function $h : \mathbb{N} \to \mathbb{N} \to \mathbb{T}$ using the BAC_ω, such that:

$$\forall i : \mathbb{N}.\ \mathcal{M} \vDash_{h(\pi_1 i, \pi_2 i);\rho} \varphi_i \to \mathcal{M} \vDash_\rho \dot{\forall}\varphi_i,$$

where π_1 and π_2 are the projection of the Cantor pairing, which exhausts all the environments of h.

A new environment ρ_s by merging the original environment with all the witnesses using the parity merge operation is defined as follows:

$$\rho_s(2n) = \rho(n)$$
$$\rho_s(2n+1) = h(\pi_1\ n, \pi_2\ n)$$

The environment ρ_s incorporates all the witnesses, allowing us to reason about the truth of quantified formulas in the original environment ρ.

Therefore, for any given environment $\rho : \mathbb{N} \to \mathcal{M}$, the environment $\rho_s : \mathbb{N} \to \mathcal{M}$ exists, s.t. the relation $I\ \rho\ \rho_s$ holds. ☐

Definition 7 (ι)**.** As I is a total relation, by applying the DC, a chain $F : \mathbb{N} \to \mathbb{N} \to \mathcal{M}$ can be constructed that, such that

$$\forall n.\ I(F\ n, F\ (n+1))$$

Using the Cantor pairing functions π_1 and π_2 again, we can then define a function $f : \mathbb{N} \to \mathcal{M}$ as

$$\iota(x) := F(\pi_1(x), \pi_2(x))$$

It can then be shown that ι is a fixed point of I, i.e., it satisfies $I(\iota, \iota)$, which means that ι is Henkin.

First, we define $I_l(\rho, \rho_s, \varphi)$, which is the left part of I, to mean that the Henkin witness of φ can be found in ρ_s when all free variables are taken from ρ.

$$I_l(\rho, \rho_s, \varphi) := \ (\forall m.\ \mathcal{M} \vDash_{(\rho_s\ m).:\rho} \varphi) \to \mathcal{M} \vDash_\rho \dot{\forall}\varphi$$

For a formula φ with n free variables, there exists a natural number k such that φ does not depend on any free variable with an index greater than k. In other words, the formula φ only depends on the first k free variables.

Lemma 3. For any two environments ρ and ρ', if a formula φ is bounded by k and the first k values of ρ are contained in ρ', then any environment that contains the witness of φ in ρ' also contains the witness of φ in ρ.

$$(\forall \varphi.\ I_l(\rho', \rho_s, \varphi)) \to \forall \varphi_k.\ \rho \subseteq_k \rho' \to I_l(\rho, \rho_s, \varphi_k)$$

where $\subseteq_k$ is defined by $\rho \subseteq_k \rho' := \forall x.x < k \to \exists y.\ \rho\ x = \rho'\ y$.

Proof. For any φ, since ρ' contains the first k values of ρ, there is a fixed substitution σ such that:

$$\mathcal{M} \vDash_\rho \dot{\forall}\varphi \iff \mathcal{M} \vDash_{\rho'} (\dot{\forall}\varphi)[\sigma]$$

As $I_l(\rho', \rho_s)$ holds for all ϕ, let $\phi := \varphi[\text{up}\ \sigma]$. Then there exists a witness w in ρ_s such that:

$$\mathcal{M} \vDash_{(w.:\rho')} \varphi[\text{up}\ \sigma] \to \mathcal{M} \vDash_{\rho'} (\dot{\forall}\varphi)[\sigma].$$

And w is also the witness of φ in ρ, obtained by the above two equations. Specifically:

$$\mathcal{M} \vDash_{(w.:\rho)} \varphi \to \mathcal{M} \vDash_\rho (\dot{\forall}\varphi).$$

□

By this lemma, we can now demonstrate that ι is Henkin.

Lemma 4 (Fixed Point)**.** There is an environment $\iota : \mathbb{N} \to \mathcal{M}$ obtained from DC and BDP satisfies

$$I(\iota, \iota)$$

As a consequence, we obtain that ι is Henkin.

Proof. Firstly, it is evident that $\iota \subseteq \iota$. To complete the proof, it is necessary to show that $I_l(\iota, \iota, \varphi)$ holds for all φ.

Let k be the bounded of formula φ, by applying the lemma above, $I_l(\iota, \iota, \varphi)$ can be obtained by getting an e for any k such that $f \subseteq_k F_e$ and $I_l(F_e, \iota, \phi)$ for all ϕ.

To achieve a constructive proof, we need to obtain an e by searching linearly for the first k elements such that $\forall x < k, \pi_1 x < e$. This allows us to find the eth environment F_e in the chain that includes the first k elements of ι. Applying the monotonicity of F, which is $\forall x < y, F_x \subseteq F_y$. Therefore, e is the number that satisfies this condition.

Noticed that the monotonicity of F is evident because the invariant I guarantees that the next environment always contains the previous one. Thus, it can be proven by induction.

For the second part, $\forall n, F_n \subseteq \iota$ by definition of ι. □

Theorem 4 (Löwenheim-Skolem Theorem IV)**.** Assuming DC and BDP, for any model $\mathcal{M}$ within a countable signature, there is a countable syntactic model $\mathcal{N}_\iota$ such that

$$\mathcal{N}_\iota \preceq \mathcal{M}.$$

Proof. The Lemma 4 Fixed Point shows that there is a Henkin environment ι under the assumption DC and BDP, by applying Theorem 3 Löwenheim-Skolem Theorem III, the elementary submodel $\mathcal{N}_\iota$ exists. □

6. Reversing the Direction

In this section, we reverse the direction and explore the proofs of DC and BDP starting from the DLS on countable cardinality. Furthermore, we consider DLS in arbitrary dimensions to prove the BAC for arbitrary cardinalities.

The proof from DLS to DC was based on Asaf Karagila's work in 2014 [**?**]. Here we discuss the proof in constructive logic and, to avoid using LEM, we consider the full syntax, which includes the existential quantifier.

Lemma 5. For any total binary relation $R : A \to A \to \mathfrak{P}$, under the assumption of DLS, there is a function $f : \mathbb{N} \to A$, s.t.

$$\forall n.\ \exists m.\ R(fn, fm)$$

Proof. Consider a signature that only includes a predicate B, and define a model $\mathcal{B}$ over domain A, where B is interpreted on $\mathcal{B}$ as R, i.e. $B^{\mathcal{B}} := R$.

We apply DLS on the model $\mathcal{B}$, yielding a countable model $\mathcal{N}$ and an elementary embedding h. Define $f\ n := h\ (E_{\mathcal{N}}\ n)$, where $E_{\mathcal{N}} : \mathbb{N} \to \mathbb{T}$ is a computable function that enumerates all terms in $\mathcal{N}$ since $\mathcal{N}$ is countable.

By the property of elementary embeddings, $\mathcal{N} \models \dot\forall\ \dot\exists\ B$. Therefore, for any n, there exists an element $f\ m$ in $\mathcal{N}$ for which $R(fn, fm)$, since $\mathcal{N} \models_\rho B$ if and only if $\mathcal{B} \models_{h \circ \rho} B$.

□

This means that for any total binary relation, we have found a countable subset on which R still maintains totality.

Lemma 6. For any binary relation $R : A \to A \to \mathfrak{P}$,

$$\mathsf{DLS} \land R \text{ is deciadable} \to \mathsf{DC} \text{ on } R$$

Proof. If R is decidable, then by defining the Linear Search Type [**?**], we can convert the existential type on a discrete type to a Sigma type, which is a computable type. Therefore, we have $H : \forall x.\ \Sigma y.\ R(fx, fy)$. For any given $a : A$, let

$$F\ 0 := a$$

$$F\ (S\ n) := \pi_1(H(E_{\mathcal{N}}^{-1}(F\ n))),$$

where π_1 is the eliminator for Sigma Type.

It's easy to see $F(0) = a$ and $R(F(n), F(S\ n))$ for all n.

□

However, the assumption of decidable is strong, so it is possible to prove DC when R is definite, i.e. satisfies $\forall x\ y.\ R(x, y) \lor \neg R(x, y)$. Under this assumption, we can prove the relational version of DC, denoted as $\mathsf{DC}_{\mathsf{pred}}$.

Definition 8 (Relational Dependent Choice)**.** For any total binary relation R, there is a functional and total relation $F : \mathbb{N} \to A \to \mathfrak{P}$, s.t.

$$F\ 0\ r\ \land\ \forall nxy.\ (F\ n\ x \land F\ (Sn)\ y \land\ R\ x\ y).$$

Lemma 7. For any binary relation $R : A \to A \to \mathfrak{P}$,

$$\mathsf{DLS} \land R \text{ is defined} \to \mathsf{DC}_{\mathsf{pred}} \text{ on } R$$

Proof. The first step is to apply Lemma 5. Since R is definite, for any proposition $\exists y.\ R\ x\ y$, there is always exists a minimal y (but not computable), i.e., $\min\{y|R\ x\ y\}$. Let F be defined as $\lambda\ n\ y.\ y = \min\{y|R\ (f\ n)\ y\}$. It's easy to verify that F satisfies all the properties mentioned above. □

Assuming the Axiom of Unique Choice, we can obtain DC from $\mathsf{DC}_{\mathsf{pred}}$.

Based on those two poofs, we can see that the role of DLS is only to preserve any property to a countable subset. That is, choices that are more refined than countable cannot be made.

The other part is that the BDP can be obtained from the DLS.

Lemma 8. $\mathsf{DLS} \to \mathsf{BDP}$.

Proof. Consider any predicate $P : A \to \mathfrak{P}$, and let $\mathcal{U}$ be a model with only a unary predicate $P^{\mathcal{U}}$ over the domain A. By applying the LS theorem to U, we obtain an elementary embedding h from a countable model N to $\mathcal{U}$. We define $I := h \circ E_{\mathbb{T}}$, where $E_{\mathbb{T}}$ is any function that enumerates the domain $\mathbb{T}$ of terms of $\mathcal{N}$, and I is the function needed in BDP that contains the witness.

Applying the property of elementary embedding, we have:

$$(\forall x.\ P\ x := \mathcal{U} \models_{\rho \circ h} \dot{\forall} P) \iff \mathcal{N}_h \models_{\rho} \dot{\forall} P$$

Unfolding the semantics of $\dot{\forall}$ for right side to get:

$$\forall t : \mathbb{T}.\ \mathcal{N}_h \models_{t;\rho} P,$$

which is the same as for all $n : \mathbb{N}$

$$P(I\ n) := \mathcal{M} \models_{I(n);\rho} P,$$

by applying the property of elementary embedding again with P. □

Based on the available results [?], in classical logic, DLS implies AC_ω, Here, we consider the blurred axiom of choice in any cardinality κ here.

Lemma 9. $\mathsf{DLS} \to \mathsf{BAC}_\omega$

Proof. For any total relation $R : \mathbb{N} \to A \to \mathfrak{P}$, consider countably many unary predicates P_n, and define a model $\mathcal{U}$ on domain A with interpretation $P_n^{\mathcal{U}} a := R\ n\ a$.

Applying the LS theorem yields an elementary embedding $h : \mathcal{N} \to \mathcal{U}$. Since $\mathcal{N}$ is countable, there exists an iterate function $E_{\mathcal{N}}$ on $\mathcal{N}$, and we define $fn := h(E_{\mathcal{N}} n)$. It can be seen that, due to the property of elementary embeddings, $\dot{\exists} y.P_n\ y$ holds for $\mathcal{N}$, and therefore there exists an element y in $\mathcal{N}$ that satisfies $P_n y$ for all n. This verifies the property:

$$\forall n.\ \exists m.R\ n\ (f\ m)$$

. □

The proof can be applied to any cardinality at least as large as countably infinite.

For any such cardinality κ, we define LS_κ to be the statement that, for signatures with at most κ many symbols, there exists an elementary submodel of cardinality κ.

Similarly, BAC_κ is the statement that for any type B of cardinality at most κ and any total relation $R : B \to A \to \mathfrak{P}$, there exists a function $f : B \to A$ such that $\forall x.\ \exists y.\ R\ x\ (f\ y)$.

Lemma 10. $\mathsf{LS}_\kappa \to \mathsf{BAC}_\kappa$

Proof. The proof follows the same steps as in the countable case, except that we replace $\mathbb{N}$ with a type A with cardinality κ. □

7. BDP and BAC

In this section, we discuss the relationship between BDP and other related axioms.

The use of BDP allows for the selections of countable objects and ensures that a Henkin witness is among them, the definition can be extended to encompass arbitrary cases, not just countable ones.

Definition 9 ($\mathsf{BDP}^\beta_\alpha$ and $\mathsf{BDP}'^\beta_\alpha$)**.** A blurred form of Drinker Paradox $\mathsf{BDP}^\beta_\alpha$ is defined over type α and β is:

$$\forall R : \beta \to \mathfrak{P}.\ \exists f : \alpha \to \beta.\ (\forall a.\ P\ (f\ a)) \to \forall x.\ P\ x.$$

A convenient shorthand is to use $\mathsf{BDP}_\alpha := \forall \beta.\ \mathsf{BDP}^\beta_\alpha$.

Thus, the previously defined BDP corresponds to $\mathsf{BDP}_\mathbb{N}$, but this notation allows us to consider the axiom for arbitrary ordinals.

Also, the dual form $\mathsf{BDP}'^\beta_\alpha$ is defined as follow:

$$\forall R : \beta \to \mathfrak{P}.\ \exists f : \alpha \to \beta.\ (\exists x.\ P\ x) \to \exists a.\ P\ (f\ a).$$

Now in this gerneral setting, we show the relation between BDP and LEM.

Lemma 11. For any type α, there is

$$\mathsf{BDP}_\alpha + \mathsf{BDP}^\alpha_\mathbb{I} \iff \mathsf{DP} \iff \mathsf{LEM}$$
$$\mathsf{BDP}'_\alpha + \mathsf{BDP}'^\alpha_\mathbb{I} \iff \mathsf{DP}' \iff \mathsf{LEM},$$

where $\mathbb{I}$ is the unit type within a single element.

Additional, it can be proven that they are duals of each other, i.e., for the negation versions of them:

Definition 10.

$$\mathsf{NBDP} := \forall P : A \to \mathfrak{P}.\ \exists f : \mathbb{N} \to A.\ (\forall n.\ \neg P(f\ n)) \to \forall x.\ \neg Px$$
$$\mathsf{NBDP}' := \forall P : A \to \mathfrak{P}.\ \exists f : \mathbb{N} \to A.\ (\exists x.\ \neg Px) \to \exists n.\ \neg P(f\ n)$$

Fact 5.

$$\mathsf{BDP} \to \mathsf{NBDP}$$
$$\mathsf{BDP} \to \mathsf{NBDP}$$

For BAC_κ, there is also a similar situation, that is, BAC_κ narrows the selection scope to a cardinality of κ, but does not provide a function. To supplement this, if we define an AC for serving κ, it is called BAC^c_κ.

Definition 11 (BAC^c_α)**.** For type α, if the relation $R : A \to A \to \mathfrak{P}$ is total, there is a function $f : A \to A$, such that

$$\forall n.\ R\, n\, (f\, n).$$

Fact 6.

$$\mathsf{BAC}^c_\kappa + \mathsf{BAC}_\kappa \iff \mathsf{AC}_\kappa$$

It can be seen that even if BAC_κ holds for any cardinality κ, AC cannot be obtained.

8. Conclusion

In our proof of DLS, we offer a new perspective on the construction of submodels and the necessary elements required for their construction. Our approach considers all formulas and does not modify the signatures or construct distinct submodel objects. Instead, we emphasize the importance of carefully considering the environment in which the submodel is constructed.

Additionally, we have examined the proof of DLS within constructive logic. In doing so, we have demonstrated that both BDP and DC are used and have shown that BAC_κ can be derived from LS_κ in constructive logic.

The final table presents our results and compares them with those obtained in classical proofs:

Cardinality	Classical Logic	Constructive Logic
$\aleph_0$	$\mathsf{DC} \iff \mathsf{DLS}$	$\mathsf{DC} + \mathsf{BDP} \to \mathsf{DLS}$ $\quad$ $\mathsf{DLS} \to \mathsf{BDP}$
Any κ^2	$\mathsf{AC}_\kappa + \mathsf{DC} \iff \mathsf{LS}_\kappa$	$\mathsf{LS}_\kappa \to \mathsf{BAC}_\kappa$

Additional, there are following facts about DC in constructive logic:

$$\mathsf{DLS} \wedge R \text{ is deciadable} \to \mathsf{DC} \text{ on } R$$
$$\mathsf{DLS} \wedge R \text{ is definite} \to \mathsf{DC}_{\mathsf{pred}} \text{ on } R$$

As can be observed from the table, there are still some remaining gaps in our results that require further investigation. Specifically, we aim to deepen our understanding of the role played by DC within constructive logic, as well as to explore the strengths and limitations of BDP and DP.

(Haoyi Zeng) Saarland University