# The Downward Löwenheim-Skolem Theorem and the Blurred Drinker Paradox

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# Introduction

**Downward Löwenheim-Skolem Theorem**

For any infinite model $\mathcal{M}$ over a countable signature, there exists a countable submodel.

- Skolem (1920) : Axiom of Choice (AC) implies $\mathsf{LS}_{\downarrow}$
- Bunn (1984); Boolos et al. (1989): Axiom of Dependent Choice (DC) equivalent to $\mathsf{LS}_{\downarrow}$.
- This talk: Reexamine this equivalence from the perspective of constructive reverse mathematics.



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- Bunn (1984); Boolos et al. (1989): Axiom of Dependent Choice (DC) equivalent to $\mathsf{LS}_{\downarrow}$.
- This talk: Over constructive logic assuming DC, $\mathsf{LS}_{\downarrow}$ is equivalent to a **Blurred form of Drinker Paradox** (BDP).



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# Overview

**PART A**

- Weaker conclusions
- Stronger assumptions
- DC + BDP $\rightarrow$ $LS_{\downarrow}$

**PART B**

- Reversing the direction

**PART C**

- Remarks on BDP

# First Order Logic

**Definition (Syntax, c.f. Kirst et al. (2022))**

Represented as inductive type over signature $(\mathcal{F}_\Sigma, \mathcal{P}_\Sigma)$, symbols $f : \mathcal{F}_\Sigma$ and $P : \mathcal{P}_\Sigma$.

$$t : \mathbb{T} ::= x_n \mid f\ \vec{t} \quad (n : \mathbb{N})$$

$$\psi,\ \varphi : \mathbb{F} ::= \dot{\bot} \mid P\ \vec{t} \mid \psi \dot{\to} \varphi \mid \dot{\forall} \varphi \qquad \text{(Fragment Syntax)}$$

$$\psi,\ \varphi : \mathbb{F}^* ::= \dot{\bot} \mid P\ \vec{t} \mid \psi \dot{\to} \varphi \mid \dot{\forall} \varphi \mid \psi \dot{\land} \varphi \mid \psi \dot{\lor} \varphi \mid \dot{\exists} \varphi \qquad \text{(Full Syntax)}$$

Note: $\mathbb{F}^c$ and $\mathbb{T}^c$ represent all closed terms and formulas respectively.

# First Order Logic

**Definition (Semantics)**

A (Tarski) model $\mathcal{M}$ over a domain $M$ is a family of functions

$$f^{\mathcal{M}} : M^{|f|} \to M \qquad P^{\mathcal{M}} : M^{|P|} \to \mathfrak{P}$$

**Definition (Environment)**

Environment $\rho : \mathbb{N} \to \mathcal{M}$ are recursively extended to term evaluations $\hat{\rho} : \mathbb{T} \to \mathcal{M}$.

**Substitution:**

$$\varphi[t] \text{ by } t : \mathbb{T} \qquad \varphi[w] \text{ by } w : \mathcal{M}$$

**Satisfiable:**

$$\mathcal{M} \vDash \varphi := \forall \rho.\, \mathcal{M} \vDash_\rho \varphi$$

# PART A
Weaker conclusions

# Weaker conclusions

Skolem (1922): Proof of $\mathrm{LS}_{\downarrow}$ that do **not** rely on any choice principle.

Recap: The Henkin-proof of completeness theorem, c.f Henkin (1949); Herbelin and Ilik (2016); Forster et al. (2021):
There is a **syntactic model** $\mathcal{N}$ for any consistent theory.

# Weaker conclusions

- Terms type $\mathbb{T}$ is **countable** if signature is.
- The theory of classical model $\mathsf{Th}(\mathcal{M})$ is a **consistent** theory.
- Syntactic model $\mathcal{N}$ is a **countable model** that has the same theory as $\mathcal{M}$.

## Löwenheim-Skolem Theorem I

For any classical model $\mathcal{M}$ with a countable signature, there is a countable syntactic model $\mathcal{N}$ such that any closed formula $\psi : \mathbb{F}^c$ satisfies

$$\mathcal{M} \vDash \psi \iff \mathcal{N} \vDash \psi.$$

This version of the $\mathsf{LS}_{\downarrow}$ theorem is preferred in most mechanized proofs (e.g. Mizar Caminati (2010) and Isabelle/HOL Blanchette and Popescu (2013)).



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# PART A

Stronger assumptions

# Stronger assumptions

## Syntactic model

For all $\psi : \mathbb{F}^c$
$$\mathcal{N} \vDash \psi \iff \mathcal{M} \vDash \psi.$$

## Syntactic **elementary** model

There is an embedding $h : \mathcal{N} \to \mathcal{M}$, such that for all $\varphi : \mathbb{F}$:

$$\mathcal{N} \vDash_\rho \varphi \iff \mathcal{M} \vDash_{h \circ \rho} \varphi$$

denote by:
$$\mathcal{N} \preceq_h \mathcal{M}$$

# Stronger assumptions

## Henkin witness

A point $w : \mathcal{M}$ is called (universal) Henkin witness for $\varphi$ if:

$$\mathcal{M} \vDash_\rho \varphi[w] \to \mathcal{M} \vDash_\rho \forall x.\ \varphi\ x$$

## Definition (The witness property)

A model $\mathcal{M}$ satisfies the witness property if the Henkin witness of any formula $\varphi : \mathbb{F}$ can be denoted by a closed term $t$, formally:

$$\exists t : \mathbb{T}^c.\ \mathcal{M} \vDash \varphi[t] \to \mathcal{M} \vDash \forall x.\ \varphi\ x.$$

# Stronger assumptions

## Definition (Syntactic Model)

For any function $i : \mathbb{N} \to \mathcal{M}$, the syntactic model $\mathcal{N}_i$ is defined by

$$f^{\mathcal{N}_i}\ \vec{t} := f\ \vec{t} \qquad P^{\mathcal{N}_i}\ \vec{t} := \mathcal{M} \vDash_i P\ \vec{t}.$$

## Theorem (Löwenheim-Skolem Theorem II)

For any model $\mathcal{M}$ with a function $i : \mathbb{N} \to \mathcal{M}$, if $\mathcal{M}$ satifies the witness property, then there is a elementary embedding from the syntactic model $\mathcal{N}_i$ to $\mathcal{M}$:

$$\mathcal{N}_i \preceq_{\hat{i}} \mathcal{M}$$

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This is a standard result, e.g. in textbook Smullyan (1996). Similarly, there is a proof based on stronger assumptions in mathlib of Lean, where the existence of the Skolem function is assumed.

# Stronger assumptions

When the witness property fails:

- Skolemization: Expand the **submodel** until all Henkin witnesses are there
- Henkinization: Expand the **signature** until the witness property is established
- Our approach: Expand the **environment**

## PART A

$\mathsf{DC} + \mathsf{BDP} \rightarrow \mathsf{LS}_{\downarrow}$

# Henkin Environment

The environment that includes all Henkin witnesses.

## Definition (Henkin Environment)

An environment $\rho : \mathbb{N} \to \mathcal{M}$ is called Henkin environment if for all formulas $\varphi : \mathbb{F}$:

$$(\forall n : \mathbb{N}.\ \mathcal{M} \vDash_\rho \varphi[x_n]) \to \mathcal{M} \vDash_\rho \forall x.\ \varphi\ x.$$

## Theorem (Löwenheim-Skolem Theorem III)

For any model $\mathcal{M}$, if the environment $\iota$ is Henkin, then

$$\mathcal{N}_\iota \preceq_{\hat{\iota}} \mathcal{M},$$

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# Construction

How to construct a Henkin environment?

- Begin: An initial environment $\rho_0$.
- For any formulas $\varphi$, figure out the **Henkin witness** $w_\varphi$.
- Add these $w_\varphi$ to the environment to get the new environment $\rho_1$.
- Based on the new environment $\rho_1$, collect all the **Henkin witnesses** $w'_\varphi$ again.
- Add these $w'_\varphi$ to the environment to get the new environment $\rho_2$.
- Describes an infinite process to construct $\rho_n$.
- Iterate until reaching a **fixed environment** that incorporate all the Henkin witnesses!

# Construction

Take an arbitrary formula $\varphi$, how to get the Henkin witness $w$, s.t.

$$\mathcal{M} \vDash_\rho \varphi[w] \to \mathcal{M} \vDash_\rho \forall x.\ \varphi\ x$$

Drinker paradox!
There is a person ($w$), such that if this person is drinking ($P\ w$), then everyone drinks ($\forall x.\ P\ x$).
For any type $A$ and predicate $P$ over $A$.

$$\exists w. P\ w \to \forall x.\ Px.$$



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# Construction

Assuming the Drinker paradox, there is a Henkin witness $w$.

But, we only need $w$ to be hidden inside our environment, i.e., $w$ is included inside a **countable blur**: $\mathbb{N} \to \mathcal{M}$.

# Construction

Therefore, we only need a Blurred form of the Drinker Paradox, which we believe to be strictly weaker than the full drinker paradox.

**Axiom (Blurred Drinker Paradox)**

$$\forall A.\, \forall P : A \to \mathfrak{P}.\, \exists b : \mathbb{N} \to A.\, (\forall n.\, P\,(b\,n)) \to \forall x.\, P\,x.$$



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# Construction

**Definition ($\rightsquigarrow$)**

Define the relation $\rightsquigarrow : (\mathbb{N} \to \mathcal{M}) \to (\mathbb{N} \to \mathcal{M}) \to \mathfrak{P}$.

$$\rho \rightsquigarrow \rho_s := \forall \varphi.\ (\forall m.\ \mathcal{M} \vDash_\rho \varphi[\hat{\rho}_s\ m]) \to \mathcal{M} \vDash_\rho \dot{\forall} \varphi \ \wedge\ \rho\ \subseteq \rho_s$$

If an environment $\rho$ is a fixed point of $\rightsquigarrow$, s.t. $\rho \rightsquigarrow \rho$, then $\rho$ is Henkin.

# Construction

DC is a principle in mathematical logic that is strictly weaker than the Axiom of Choice.

**Axiom (Dependent Choice (DC))**

For any binary total relation $R : A \to A \to \mathfrak{P}$,

$$\exists f : \mathbb{N} \to A. \forall n.\ R\ (f\ n)\ (f\ (n+1)).$$

Countable Choice can be obtained from DC.

**Definition (Countable Choice ($\mathsf{AC}_\omega$))**

For any total relation $R : \mathbb{N} \to A \to \mathfrak{P}$ over a countable set, there is a function $f : \mathbb{N} \to A$, s.t.

$$\forall n.\ R\ n\ (f\ n).$$



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# Construction

**Theorem (Totality of $\rightsquigarrow$)**

For any environment $\rho$, there is an environment $\rho_s$, s.t.

$$\rho \rightsquigarrow \rho_s$$

For any environment $\rho$:

- A blurred function $h' : \mathbb{N} \to \mathcal{M}$ for formula $\varphi$ by BDP
- A function $h : \mathbb{F} \to \mathbb{N} \to \mathcal{M}$ exhaust all formulas by $\mathrm{AC}_\omega$

Since $\mathbb{F}$ is countable:

$$h : \mathbb{N} \to \mathbb{N} \to \mathcal{M}$$

# Construction

**Theorem (Totality of $\rightsquigarrow$)**

For any environment $\rho$, there is an environment $\rho_s$, s.t.

$$\rho \rightsquigarrow \rho_s$$

**Proof.**

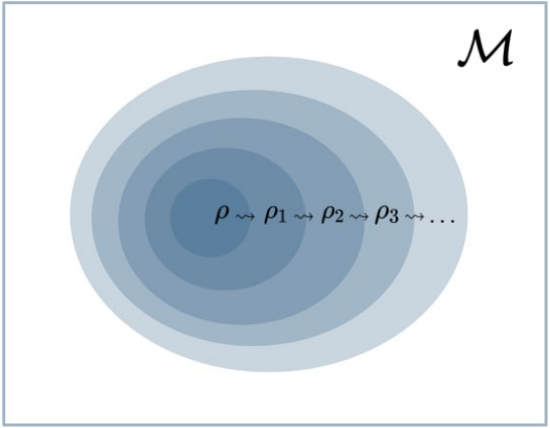
We can then obtain a function $h : \mathbb{N} \to \mathbb{N} \to \mathbb{T}$ that incorporates the Henkin witnesses using the $\mathrm{AC}_\omega$ and BDP.

$$\rho_s(2n) = \rho(n)$$
$$\rho_s(2n+1) = h(\pi_1\ n, \pi_2\ n)$$

As a result, $\exists \rho_s.\ \rho \rightsquigarrow \rho_s$. $\square$

# Construction

Applying DC on this total relation, we have a sequence of compatible environments $\rho_n : \mathbb{N} \to \mathcal{M}$ for all natural numbers $n$.



# Construction

Let $F : \mathbb{N} \to \mathbb{N} \to \mathcal{M}$ be the function obtained from DC on relation $\rightsquigarrow$.
Define an environment

$$\iota(x) := F(\pi_1\ x, \pi_2\ x)$$

**Theorem (Fixed point of $\rightsquigarrow$)**

There is an environment $\iota : \mathbb{N} \to \mathcal{M}$ obtained from DC and BDP that satisfies:

$$\iota \rightsquigarrow \iota$$

As a consequence, we obtain that $\iota$ is Henkin.

**Theorem (Löwenheim-Skolem Theorem IV)**

DC + BDP → $\mathsf{LS}_{\downarrow}$.

# PART B
Reversing the direction

# Reversing the direction

We largely follow Karagila (2014), and we are now working under the setting of full syntax.

$$\mathsf{LS}_{\downarrow} \to \mathsf{DC}$$

For any total binary relation $R : A \to A \to \mathfrak{P}$.

**Idea:** Define a model $\mathcal{A}$ over $A$ with only one binary predicate symbol $\mathsf{R}$, and $\mathsf{R}^{\mathcal{A}} := R$.
Let $\mathcal{M} \vDash \forall x.\ \exists y.\ \mathsf{R}(x, y)$, therefore, we have $\mathcal{N}_{\iota} \vDash_{\hat{\iota}} \forall x.\ \exists y.\ \mathsf{R}(x, y)$.

Assuming $R$ is decidable, we can **search over** $\mathbb{N}$, therefore, there is a (computable) sequence of witnesses $w_1, w_2, \ldots$, s.t.

$$\forall n.\ R(w_n, w_{n+1})$$

# Reversing the direction

$$\mathsf{LS}_{\downarrow} \to \mathsf{BDP}$$

**Idea:** For any predicate $P : A \to \mathfrak{P}$, let $\mathcal{M} \vDash_{\rho} \dot{\forall} P \iff \mathcal{N}_{\iota} \vDash_{\hat{\iota} \circ \rho} \dot{\forall} P$.

$$\forall w : \mathbb{T}.\, \mathcal{N} \vDash P[w] \to \forall x.\, Px.$$

The drinker hiding in the countable domain $\mathbb{T}$.

# PART C
Remarks on BDP

# Blurred form of drinker paradox

**Definition ($\mathsf{BDP}_A^B$ and $\mathsf{BDP'}_A^B$)**

General blurred form of Drinker Paradox $\mathsf{BDP}_A^B$ over types $A$ and $B$ is defined by:

$$\forall R : B \to \mathfrak{P}.\ \exists f : A \to B.\ (\forall a.\ P\ (f\ a)) \to \forall x.\ P\ x$$

Let $\mathsf{BDP}_A := \forall B.\ \mathsf{BDP}_A^B$, then $\mathsf{BDP} = \mathsf{BDP}_{\mathbb{N}}$.
Also, the dual form $\mathsf{BDP'}_A^B$ is defined as follow:

$$\forall R : B \to \mathfrak{P}.\ \exists f : A \to B.\ (\exists x.\ P\ x) \to \exists a.\ P\ (f\ a).$$

For any type $A$, there is

$$\mathsf{BDP}_A + \mathsf{BDP}_{\mathbb{I}}^A \iff \mathsf{DP} \iff \mathsf{LEM}$$
$$\mathsf{BDP'}_A + \mathsf{BDP'}_{\mathbb{I}}^A \iff \mathsf{DP'} \iff \mathsf{LEM},$$

# More about BDP

## Limited Principle of Omniscience (LPO)

$$\forall f : \mathbb{N} \to \mathbb{B}.\ (\forall x.\ f x = \textsf{false}) \lor (\exists x.\ f x = \textsf{true})$$

## Independence of Premise (IP)

$$\forall (P : A \to \mathfrak{P})(Q : \mathfrak{P}).\ A \to (Q \to \exists x.\ P x) \to \exists x.\ Q \to P\ x.$$

$$\text{LEM} \iff \text{BDP}_{\mathbb{N}} + \text{BDP}_{\mathbb{I}}^{\mathbb{N}} \iff \text{BDP}_{\mathbb{N}} + \text{LPO} \iff \text{IP}$$

$$\text{BDP}_{\mathbb{I}}^{\mathbb{N}} \to \text{LPO}$$

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[1] More results: Blurred form of the IP $\iff$ BDP′, omniscient DC implies BDP'. etc..

# Conclusion

The final table presents our results and compares them with those obtained in classical proofs:

| Cardinality | Classical Logic | Constructive Logic | |
|---|---|---|---|
| $\aleph_0$ | DC $\iff$ $\text{LS}_\downarrow$ | $\text{DC} + \text{BDP} \to \text{LS}_\downarrow$ | $\text{LS}_\downarrow \to \text{BDP}$ |

Additional, there are following facts about DC in constructive logic:

$$\text{LS}_\downarrow \wedge R \text{ is decidable} \to \text{DC on } R$$



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# Conclusion

**Contributions**

- Complementing the connection to DC with a weak classical principle BDP
- A new approach to proving $LS_{\downarrow}$ without expanding the model or signature
- To the best of our knowledge, first mechanization of complete proof from DC to $LS_{\downarrow}$ and including the reverse logical analysis (and the facts about BDP)

**Mechanization** 2500 LOC overall based on FOL library (Kirst et al. (2022))

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