Post's Problem and The Priority Method in Synthetic Computability

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Background

Synthetic Computability

Recap

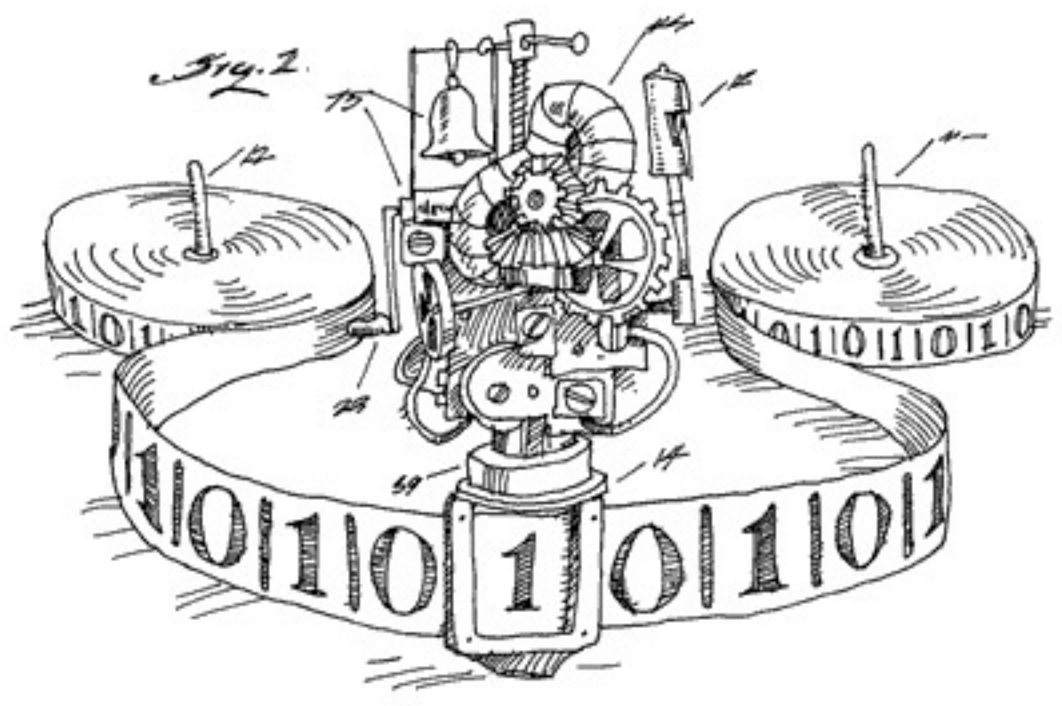
P is **Decidable**: $\exists f : X \to \mathbb{B} \,.\, P\ x \leftrightarrow f\ x = \mathsf{tt}$ $\cancel{\land\ f \text{ is computable}}$

What is computable ?

Turing Machine

λ-Calculus

Synthetic Computability



$$\begin{aligned} \mathsf{fix}\ F &:= \lambda x.\, \mathsf{fix}'\ \mathsf{fix}'\ F\ x \\ \mathsf{fix}' &:= \lambda f, F.\, F\ (\lambda x.\, f\ f\ F\ x) \end{aligned}$$

Synthetic Computability

A predicate $P : X \to \mathbb{P}$ is

Decidable $\exists\, f : X \to \mathbb{B}\,.\, P\ x \leftrightarrow\ f\ x = \mathsf{tt}$

Semi-decidable $\exists\, f : X \to \mathbb{N} \to \mathbb{B}\,.\, P\ x \leftrightarrow \exists n\,.\, f\ x\ n = \mathsf{tt}$

"Does a Turing machine halt on a given input?"

Halting Problem H

$H\ x \leftrightarrow x$-th partial function halts on x

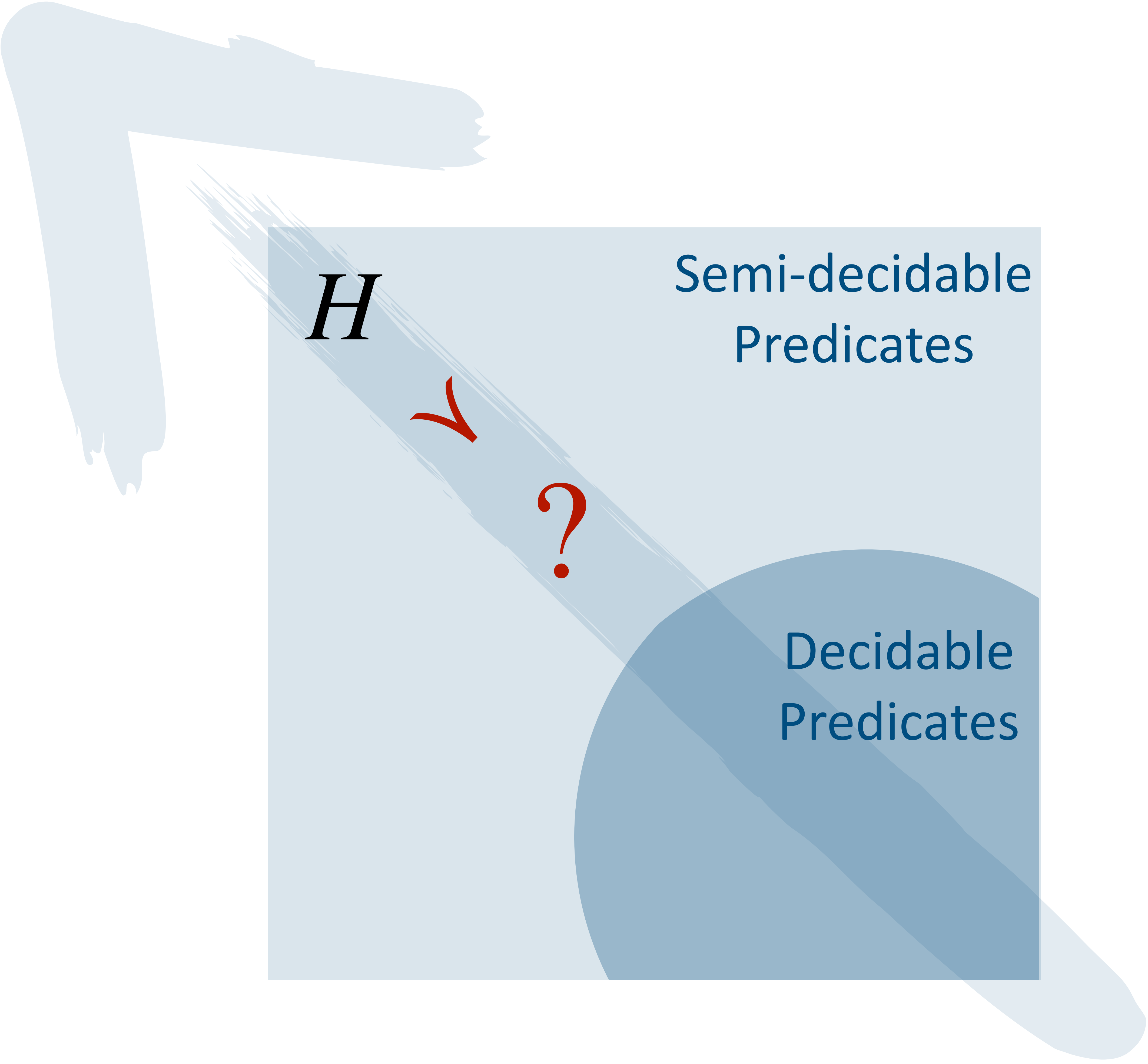
Post's Problem

"Is there an undecidable,

semi-decidable predicate

that is strictly easier than the Halting problem?"

- Post, 1944

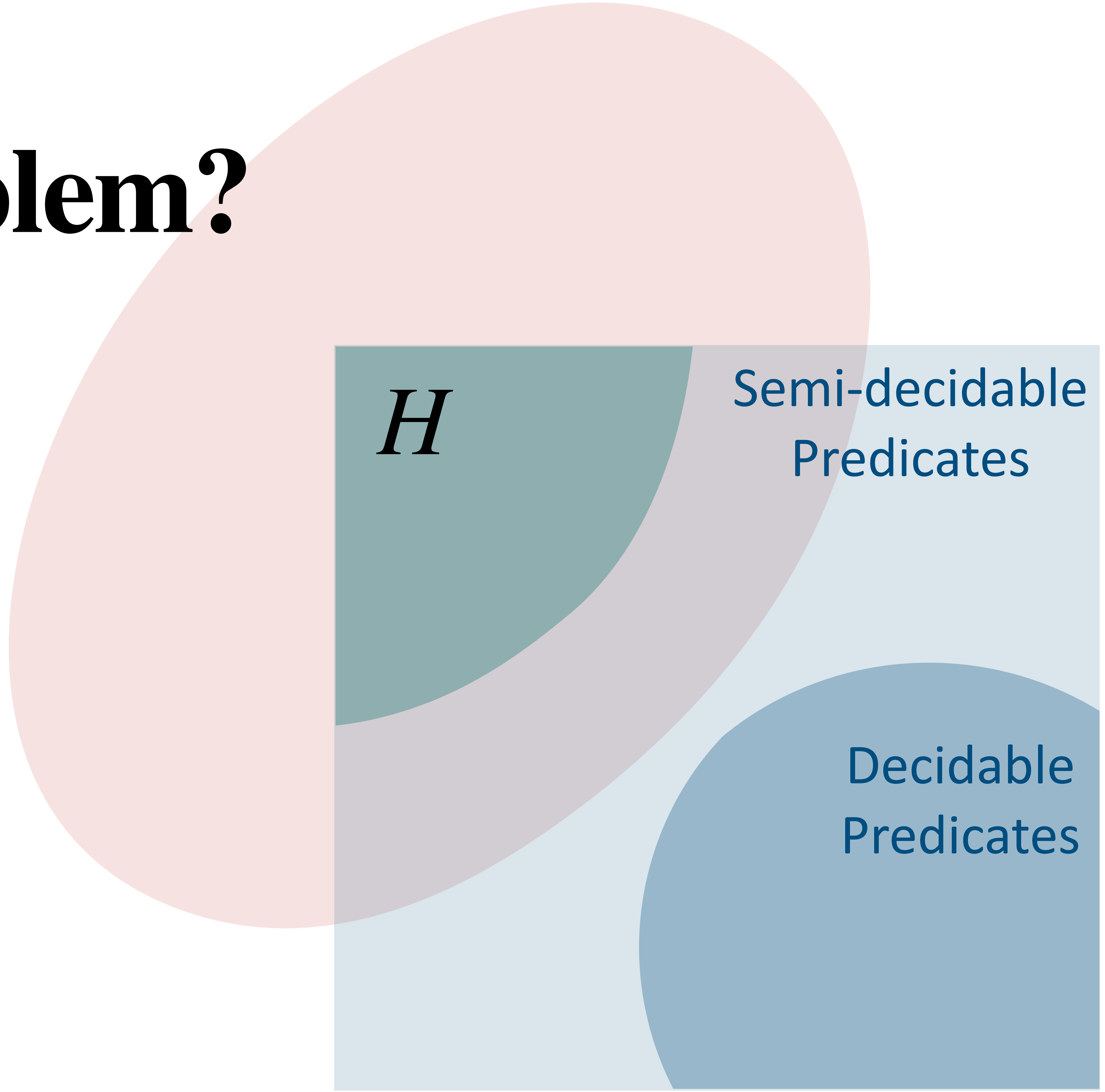


Easier than Halting Problem?

H is reducible to P

Many-one Reduction: $H \preceq_m P$

Turing Reduction: $H \preceq_T P$



Consider reductions in the most general sense, i.e., Turing reduction, which is also the problem Post left open in his paper.

Solutions to Post’s Problem

Finite extension method [Post 1944]

Simple Predicate

… **After 12 years**

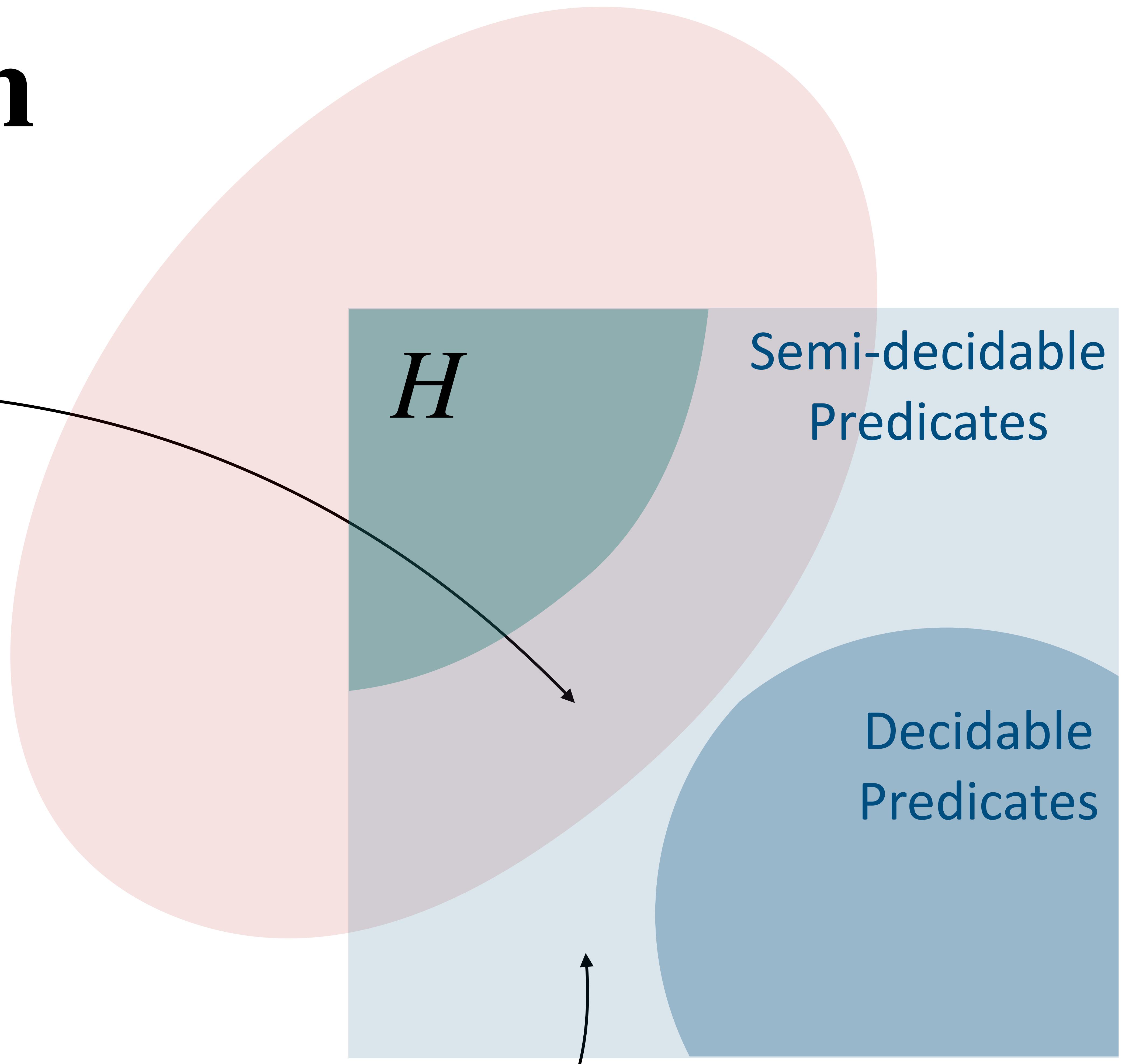
The Priority Method

Friedberg–Muchnik Theorem

[Mučnik 1956] [Friedberg 1957]

Low Simple Predicate

[Lerman & Soare 1980] [Soare 1999]

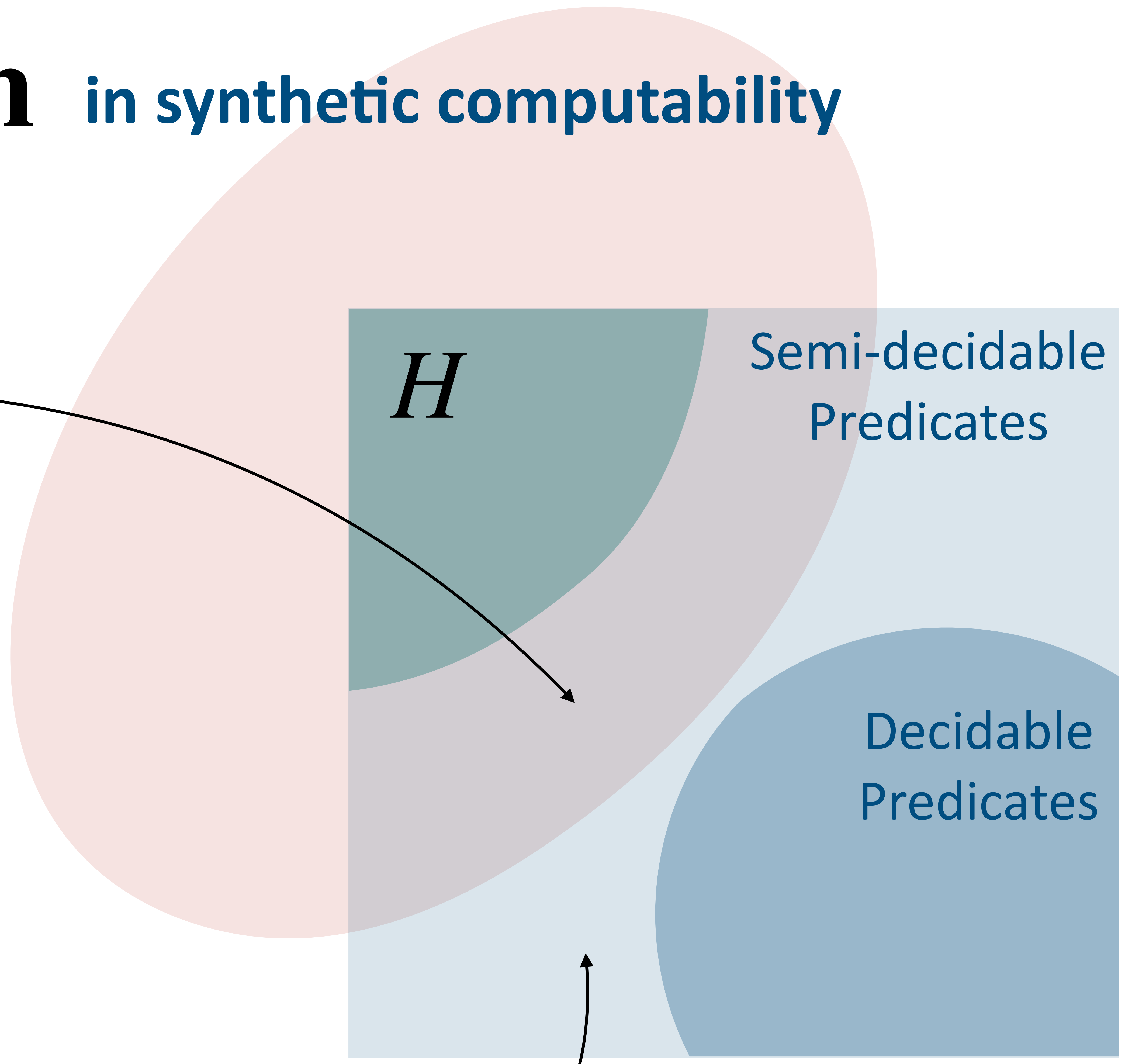


Solutions to Post's Problem in synthetic computability

in synthetic computability

[Forster & Jahn 2023]

Simple Predicate



Another 12 years ?

Low Simple Predicate

The Priority Method

The Priority Method

$$\gamma : \mathbb{N}^* \to \mathbb{N} \to \mathbb{N} \to \mathfrak{P}$$

$$\frac{}{0 \leadsto [\,]} \qquad \frac{n \leadsto L \quad \gamma_n^L\ x}{n+1 \leadsto x :: L} \qquad \frac{n \leadsto L \quad \forall x.\ \neg\ \gamma_n^L\ x}{n+1 \leadsto L}$$

The Priority Method

$$0 \rightsquigarrow \emptyset \qquad \gamma_0^{\emptyset}\ x_0 \qquad 1 \rightsquigarrow [x_0] \qquad \forall x.\ \neg\gamma_1^{[x_0]}\ x \qquad 2 \rightsquigarrow [x_0] \qquad \gamma_2^{[x_0]}\ x_1 \qquad [x_0, x_1] \qquad \dots$$

$$x \in P_\gamma \ := \ \exists n\ L.\ n \rightsquigarrow L \land x \in L$$

Let γ be an extension

γ is computable and unique $\implies$ P_γ is **semi-decidable**

Simple Extension

$$\begin{aligned} \pi_n^L\ e\ x &:= x \in \mathcal{W}_e[n] \land \omega_n^L(e) < x \\ \Pi_n^L\ e &:= L \cap \mathcal{W}_e[n] = \varnothing \land \exists x.\ \pi_n^L\ e\ x \\ \gamma_n^L\ x &:= \exists e.\ e < n \land \mathcal{L}\ e.\ \Pi_n^L \land \mathcal{L}\ x.\ \pi_n^L\ e \end{aligned}$$

$$\omega : \mathbb{N} \to \mathbb{N}^* \to \mathbb{N} \to \mathbb{N}$$

$$P_1 \prec P_2 \prec P_3 \prec P_4 \prec P_5 \prec P_6 \cdots$$

Let ω be a wall

ω is greater than $2e$ and convergent $\Rightarrow$ P_γ is **Simple**

Low Wall

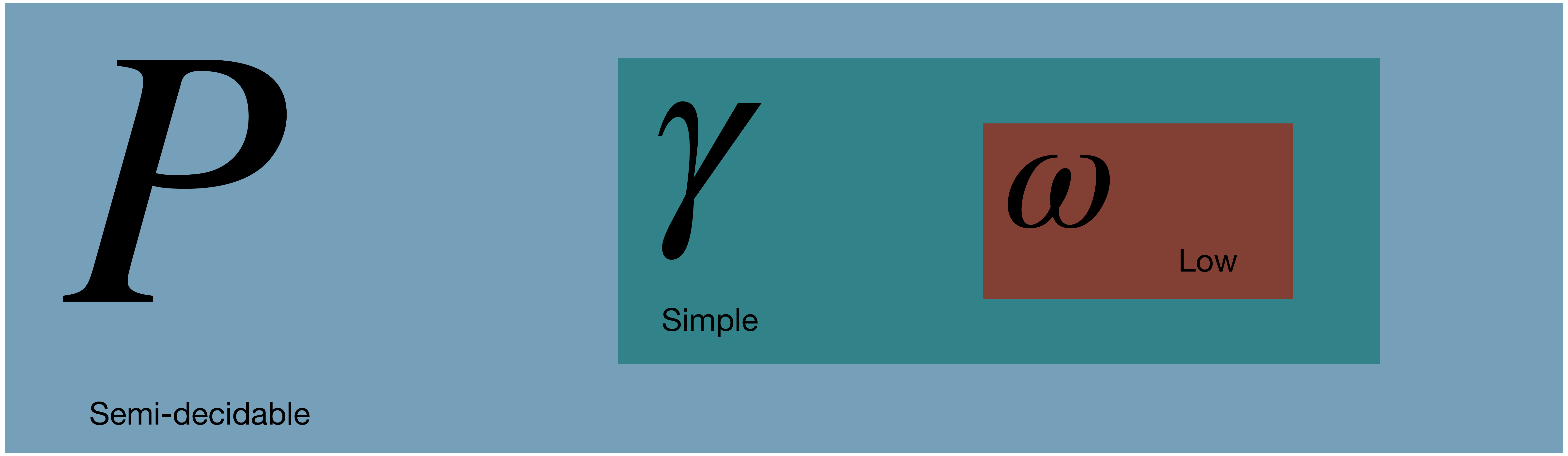
$$\omega_n^L(e) \;:=\; \max(2 \cdot e, \varphi_e^{x \in L}(e)[n]) \qquad N_e \;:=\; \exists^\infty s.\ \Phi_e^A(e)[s]\downarrow \to \Phi_e^A(e)\downarrow$$

$$P_1 \prec N_1 \prec P_2 \prec N_2 \prec P_3 \prec N_3 \cdots$$

By the Limit Lemma $\Rightarrow$ P_{γ_ω} is **Low**

Low Simple Predicate

P_{γ_ω} is **low** **simple** **predicate**



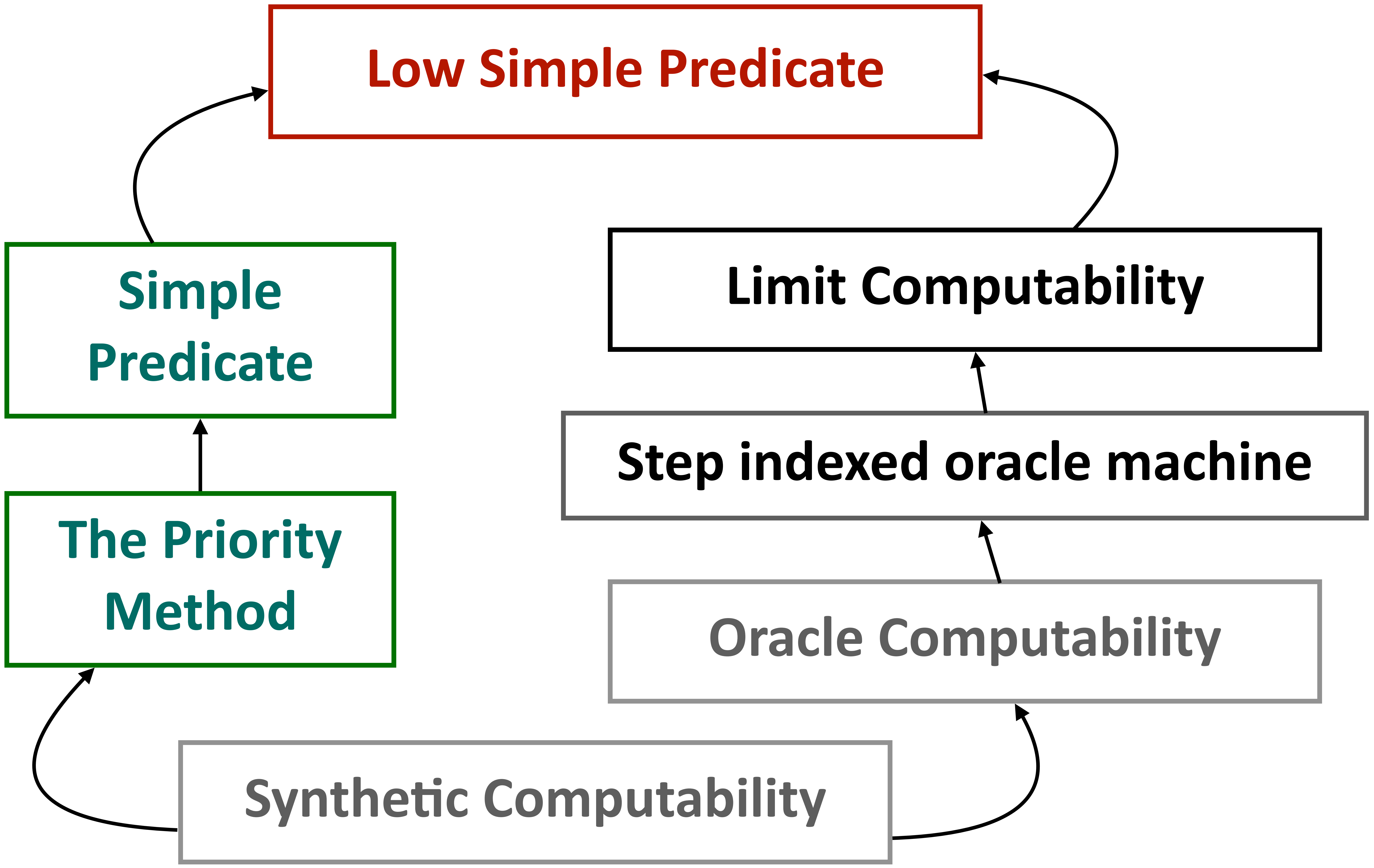
P_{γ_ω} is a positive solution to Post's Problem

Formalisation in Coq

Lines of code*: ~ 900

Lines of code: ~ 1000

Lines of code: ~ 250

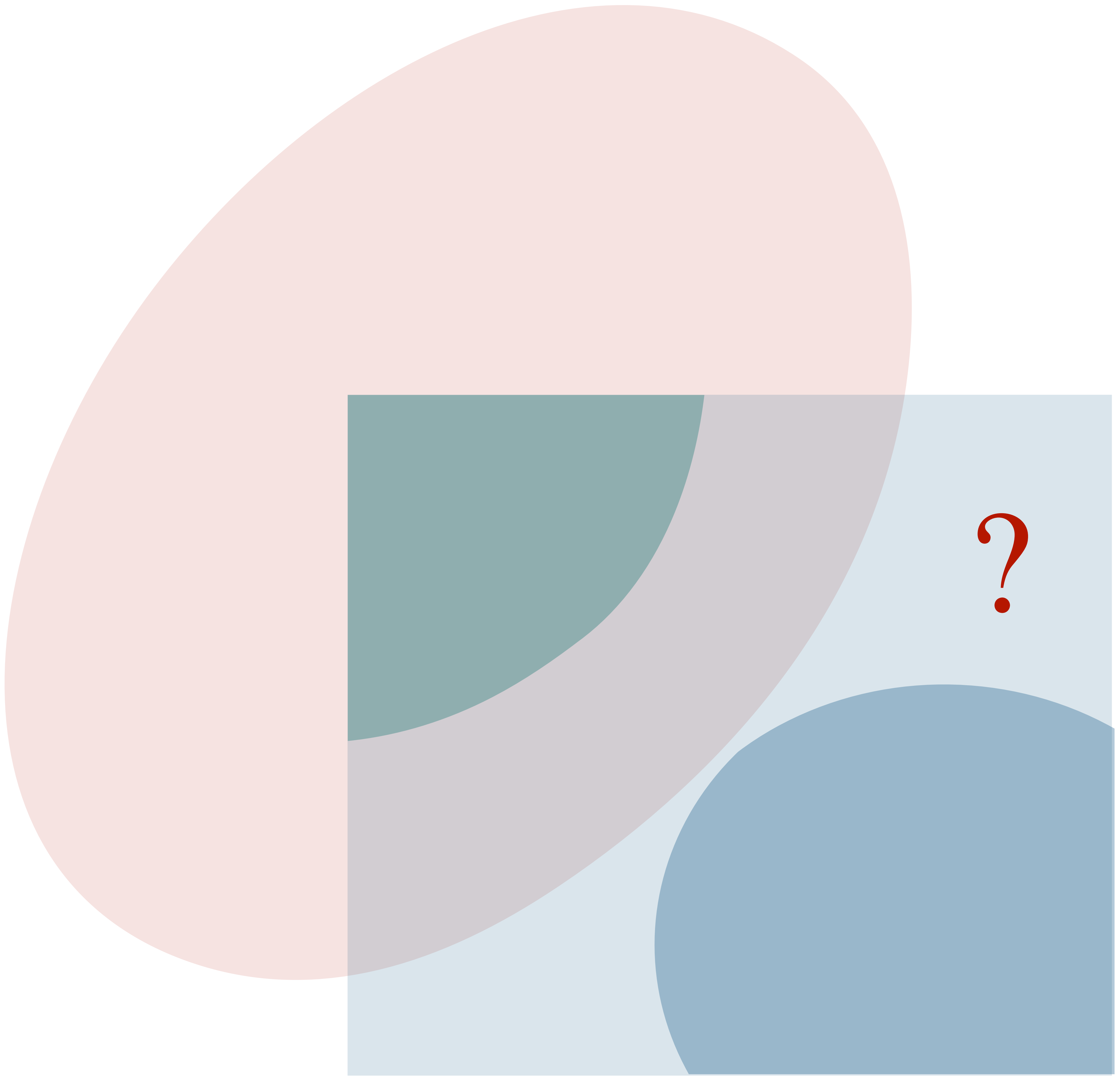


*: a technical lemma is missing

Goals (now)

We aim to show the following theorems and constructions in synthetic computability:

- Definition of limit computable
- Limit lemma
- The priority method
- Definition of low simple predicate
- Existence of low simple predicate
- Friedberg-Muchnik Theorem
- Constructive analysis



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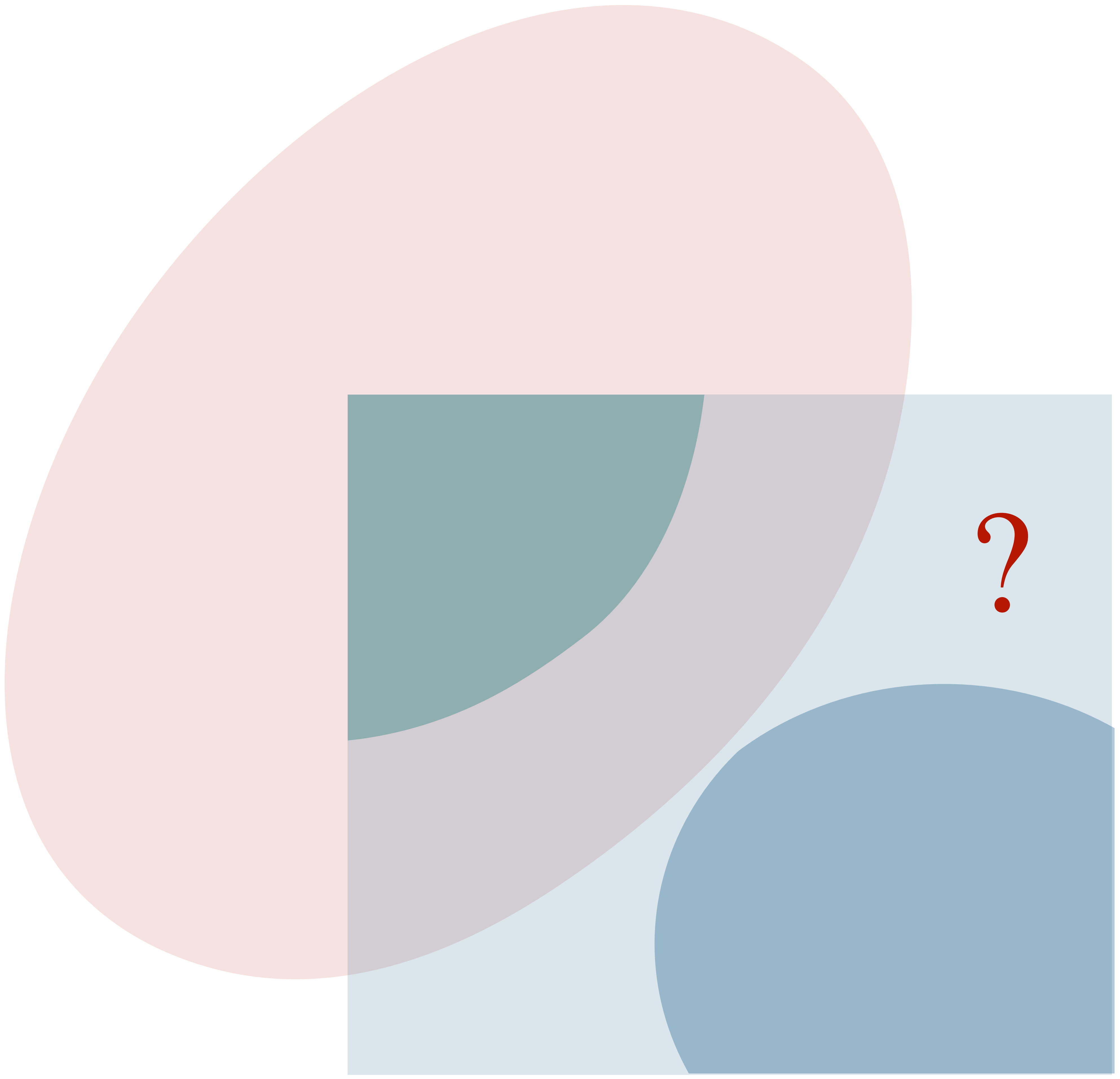

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Appendix A

Oracle Computable

Based on a notion of computability of functionals $F : (Q \to A \to \mathbb{P}) \to (I \to Q \to \mathbb{P})$, The argument $R : Q \to A \to \mathbb{P}$ is to be read as the oracle relating questions $q : Q$ to answers $a : A$, $i : I$ is the input to the computation, and $o : O$ is the output, such an F is considered (oracle)-computable if there is an underlying computation tree $\tau : I \to A^* \rightharpoonup (Q + O)$:

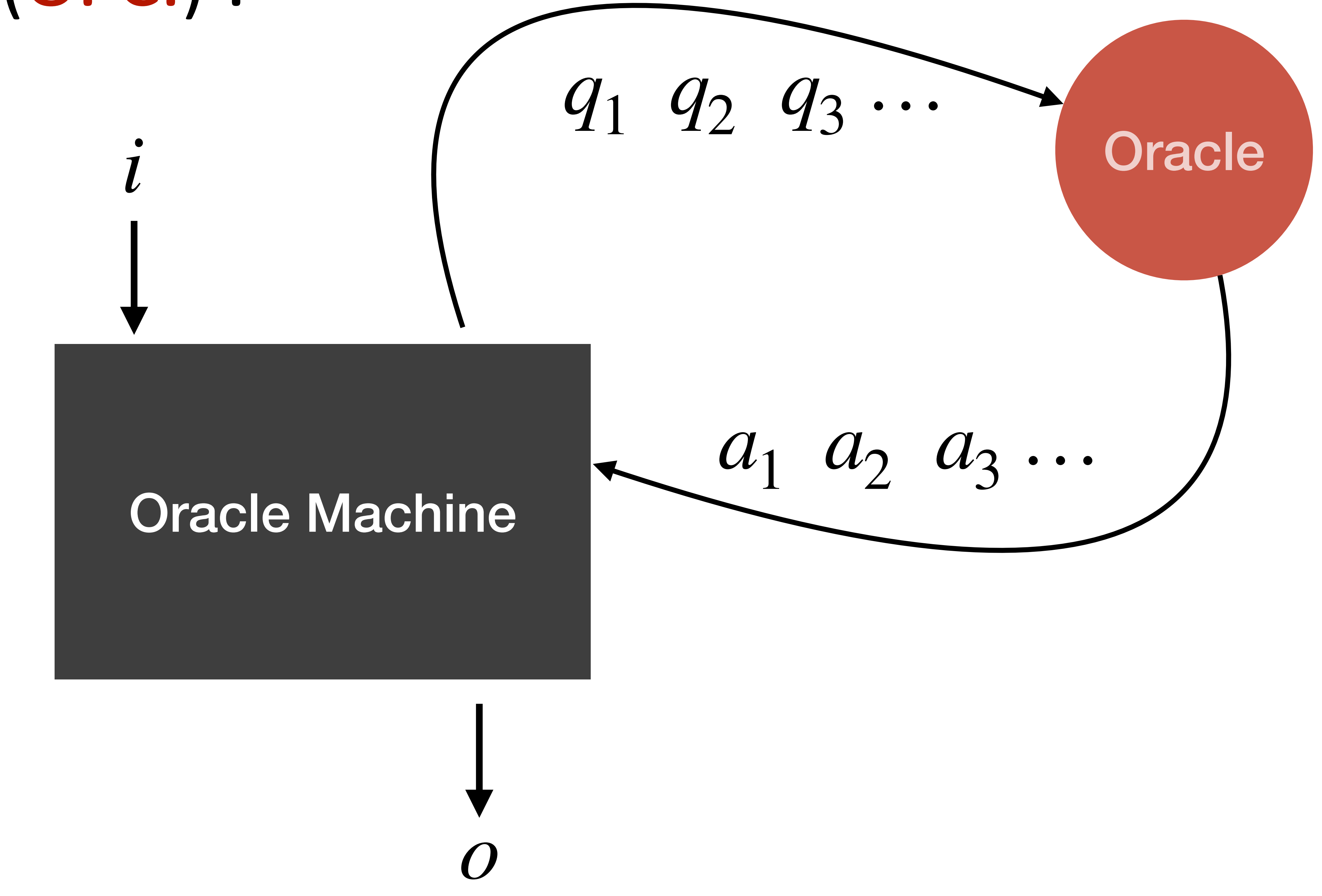
$$\forall R\ x\ b\,.\, F\ R\ x\ b \iff \exists qs\ as\,.\ \tau\ x; R \vdash qs; as \wedge \tau\ x\ as \triangleright\ \mathrm{out}\ b$$

where the interrogation relation $\sigma; R \vdash qs; as$ is inductively defined for $\sigma : A^* \rightharpoonup Q + O$ as:

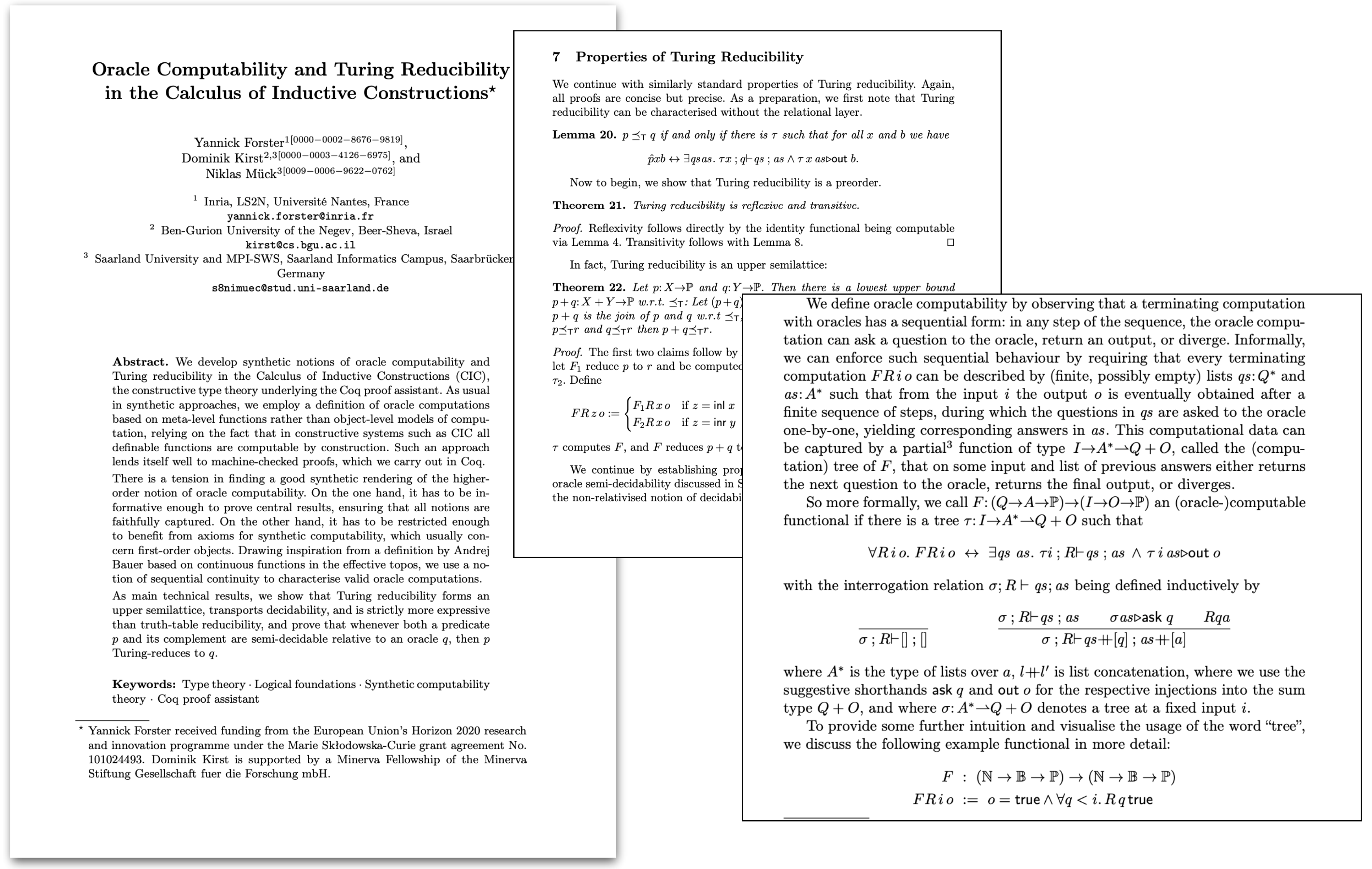
$$\frac{}{\sigma\ ; R \vdash [\,]; [\,]} \qquad \frac{\sigma\ ; R \vdash qs; as \quad \sigma\ as \triangleright \mathrm{ask}\ q \quad R(q, a)}{\sigma\ ; R \vdash qs @ [q]; as @ [a]}$$

Turing reducible in synthetic computability

Synthetic notation of Oracle Computable (O. C.) :



F is O.C. is capturing by some underlying computable object [Forster, Kirst & Mück 2023]



Turing reduction $P \leq Q := \exists F\,.\; F$ is O.C. $\wedge\ \forall x\,.\ \begin{cases} P\,x \leftrightarrow F\,\hat{Q}\,x\ \mathrm{tt} \\ \neg P\,x \leftrightarrow F\,\hat{Q}\,x\ \mathrm{ff} \end{cases}$

Appendix B

Step index function

We insert this oracle O into our Turing machine by fixing a n, and subsequently run τ. Based on this effectively computable oracle, we can define a total function Φ as follows:

$$\Phi_{\tau}^{O(n)}\ x\ i\ j\ :=\ \begin{cases} \ulcorner \mathsf{out}\ o \urcorner & \text{if } (\tau\ x\ []) \rightsquigarrow_j \mathsf{out}\ o \\ \ulcorner \mathsf{ask}\ q \urcorner & \text{if } (\tau\ x\ []) \rightsquigarrow_j \mathsf{ask}\ q \text{ and } i = 0 \\ \Phi_{\tau @ [\chi_O\ n\ q]}^{O(n)}\ x\ i'\ j & \text{if } (\tau\ x\ []) \rightsquigarrow_j \mathsf{ask}\ q \text{ and } i = S\ i' \\ \mathsf{none} & \text{otherwise} \end{cases}$$

Given that P is Turing reducible to $\varnothing$, we obtain the computable tree τ. Building upon the step-index function described above, we define the following function:

$$\chi_P(s, x)\ :=\ \begin{cases} b & \text{if } \Phi_{\tau}^{K}(x)[s] = \ulcorner b \urcorner \\ \mathsf{tt} & \text{otherwise} \end{cases}$$

Low Simple Predicate 1

undecidable predicate

Simple predicate:

If a predicate P is simple, then P is semi-decidable and $\neg(P \preceq \varnothing)$

Turing jump of P:

$P' \ x \leftrightarrow x$-th oracle machine with oracle P halts on x

Low Simple Predicate 2

Low predicate: A predicate P is low, if the Turing jump of P is reducible to K

$$P' \preceq K \Rightarrow \neg(K \preceq P)$$

Low Simple predicate: $\varnothing \prec P \prec K$, where $P \prec Q := P \preceq Q \wedge \neg(P \preceq Q)$

A positive solution to Post's Problem

Showing a predicate is reducible to K is difficult!

Use Function

$$\text{Let } k = \varphi_e^A(e)[n]$$

$$\Phi_e^p(e)[n] = \ulcorner\star\urcorner \to \forall q.\ q \equiv_k p[n] \to \Xi_e\ \hat{q}\ e\ \star$$

$$\varphi_e^p(x)[n] = k \to p[n] \equiv_k p[n+1] \to \varphi_e^p(x)[n+1] = k$$