The Blurred Drinker Paradox and Blurred Choice Axioms for the Downward Löwenheim-Skolem Theorem

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Abstract

In the setting of constructive reverse mathematics, we analyse the downward Löwenheim-Skolem (DLS) theorem of first-order logic, stating that every infinite model has a countable elementary submodel. Refining the well-known equivalence of the DLS theorem to the axiom of dependent choice (DC) over the classically omnipresent axiom of countable choice (CC) and law of excluded middle (LEM), our approach allows for several finer logical decompositions: Over CC, the DLS theorem is equivalent to the conjunction of DC with a newly identified principle weaker than LEM that we call the blurred drinker paradox (BDP), and without CC, the DLS theorem is equivalent to the conjunction of BDP with similarly blurred weakenings of DC and CC. Orthogonal to their connection with the DLS theorem, we also study BDP and the blurred choice axioms in their own right, for instance by showing that BDP is LEM without some contribution of Markov's principle and that blurred DC is DC without some contribution of CC. All results are stated in the Calculus of Inductive Constructions and an accompanying Coq mechanisation is available on Github.

Background The Löwenheim-Skolem theorem is a central result about first-order logic, entailing that the formalism is incapable of distinguishing different infinite cardinalities. In particular its so-called downward part, stating that every infinite model can be turned into a countably infinite model with otherwise the exact same behaviour, can be considered surprising or even paradoxical: even systems like ZF set theory, concerned with uncountably large sets like the reals or their iterated power sets, admit countable interpretations. This seeming contradiction in particular and its metamathematical relevance in general led to an investigation of the exact assumptions under which the downward Löwenheim-Skolem (DLS) theorem applies.

From the perspective of (classical) reverse mathematics [6, 10], there is a definite answer: the DLS theorem (for countable languages) is equivalent to the dependent choice axiom (DC), a weak form of the axiom of choice, stating that there is a path through every total relation [5, 8, 3]. To argue one direction, one can organise the countable submodel construction such that a single application of DC is needed. For the other direction, one uses the DLS theorem to turn a given total relation R into a countable sub-relation R', applies the classically provable countable choice (CC) to get a path f' through R', and reflects it back as a path f through R.

However, the classical answer is insufficient if one investigates the computational content of the DLS theorem, i.e. the question how effective the transformation of a model into a countable submodel really is. A more adequate answer can be obtained by switching to the perspective of *constructive* reverse mathematics [7, 4], which is concerned with the analysis of logical strength over a constructive meta-theory, i.e. in particular without the law of excluded middle (LEM), stating that $p \vee \neg p$ for all propositions p, and ideally also without CC [9]. In that setting, finer logical distinctions become visible and thus one can analyse whether the computational content of the DLS theorem is exactly the same as that of DC [1, 2] by investigating whether (1) it still follows from DC alone, without any contribution of LEM, and (2) whether it still implies the full strength of DC, without any contribution of CC. We show that neither (1) nor (2) is the case by observing that the DLS theorem requires a fragment of LEM, which we call the blurred drinker paradox, and that it implies only a similarly blurred fragment of DC.

The Blurred Drinker Paradox The usual drinker paradox states that in every bar there is a person such that everyone drinks if that person drinks. A blurred version is obtained by replacing that person by an at most countable subset, represented by a function with domain $\mathbb{N}$. We also introduce a blurring of the existence principle, the dual to the drinker paradox.

$$\begin{aligned}\mathsf{BDP}_A &:= \forall P : A \to \mathbb{P}. \exists f : \mathbb{N} \to A. (\forall y. P\,(f\,y)) \to \forall x. P\,x \\ \mathsf{BEP}_A &:= \forall P : A \to \mathbb{P}. \exists f : \mathbb{N} \to A. (\exists x. P\,x) \to \exists y. P\,(f\,y)\end{aligned}$$

Here $\mathbb{N}$ can be replaced by other types, e.g. with $\mathbb{1}$ one recovers the usual drinker paradoxes and, in general, larger types induce weaker principles. Also, $\mathsf{BDP}_\mathbb{N}$ and $\mathsf{BEP}_\mathbb{N}$ are provable by choosing f to be the identity function. Writing BDP if BDP_A for all inhabited A (similar for other principles), we for instance obtain that the blurred drinker paradoxes decompose LEM:

Fact 1. LEM *is equivalent to the conjunction of* BDP *(or* BEP*) and Markov's principle.*

Blurred Choice Axioms Via the same blurring technique, a version of CC can be given where the outputs of choice functions f for total relations R are hidden in a countable subset.

$$\mathsf{BCC}_A := \forall R : \mathbb{N} \to A \to \mathbb{P}. \mathsf{tot}(R) \to \exists f : \mathbb{N} \to A. \forall n. \exists m. R\,n\,(f\,m)$$

Therefore, BCC reduces CC to the special case $\mathsf{CC}_\mathbb{N}$ for relations $R : \mathbb{N} \to \mathbb{N} \to \mathbb{P}$:

Fact 2. CC *is equivalent to the conjunction of* BCC *and* $\mathsf{CC}_\mathbb{N}$*.*

Similarly, a blurred version of DC states that every directed relation contains a countable directed subrelation, where $R \circ f : \mathbb{N} \to \mathbb{N} \to \mathbb{P}$ is the pointwise composition of R with f.

$$\mathsf{DDC}_A := \forall R : A \to A \to \mathbb{P}. \mathsf{dir}(R) \to \exists f : \mathbb{N} \to A. \mathsf{dir}(R \circ f)$$

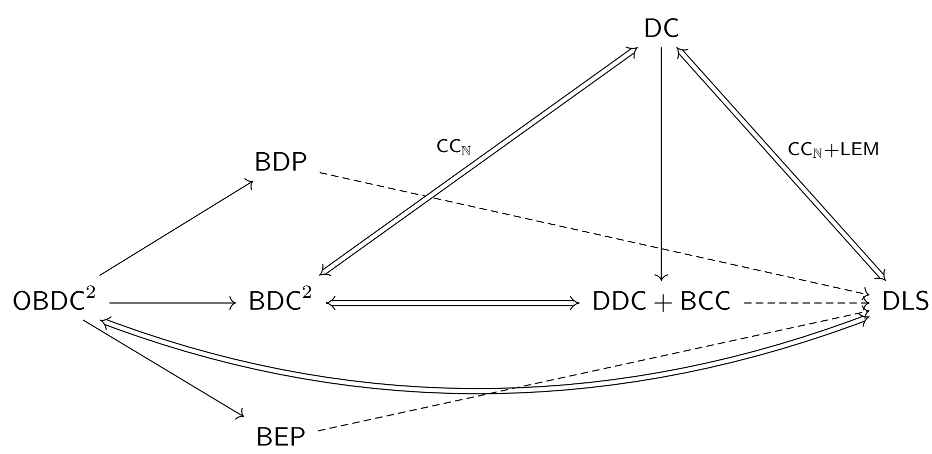
Analogous to the case of BCC, only $\mathsf{CC}_\mathbb{N}$ separates DDC from the full strength of DC:

Fact 3. DC *is equivalent to the conjunction of* DDC *and* $\mathsf{CC}_\mathbb{N}$*.*

Main Results The DLS theorem states that every first-order model $\mathcal{M}$ over a countable signature has a countable elementary submodel $\mathcal{N}$, i.e. there is an embedding $h : \mathcal{N} \to \mathcal{M}$ such that for every variable environment $\rho : \mathbb{N} \to \mathcal{N}$ and formula φ it holds that $\mathcal{N} \vDash_\rho \varphi$ iff $\mathcal{M} \vDash_{h \circ \rho} \varphi$. We obtain two logical decompositions of the DLS theorem over constructive base systems:

Theorem 1. *With* $\mathsf{CC}_\mathbb{N}$ *assumed,* DLS *is equivalent to the conjunction of* DC*,* BDP*, and* BEP*. Without any assumptions,* DLS *is equivalent to the conjunction of* BCC*,* DDC*,* BDP*, and* BEP*.*

The following diagram summarises these and further decompositions, where BDC^2 is a natural combination of DDC and BCC, and OBDC^2 further merges in BDP and BEP.



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